The University Tutorial Series.

MATRICULATION
GEOMETRY

BEING SECTIONS I–IV OF "GEOMETRY,
THEORETICAL AND PRACTICAL."

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PREFACE.

It is certain that in order to derive the full educational value from the study of Elementary Geometry, the student must be trained in both the theory and the practice of the subject. These two sections of the work are in no way antagonistic, each tends to illustrate and explain the other, and the value of either is seriously diminished whenever it loses touch with the other In the teaching of the elements of any Science, a close relation between the theoretical and practical work is of the greatest importance, but in no other case can this close relation be so easily attained as in the study of Plane Geometry

It is the object of this book to provide a course of Geometry in which theoretical and practical work are properly co ordinated. Accordingly these two parts of the subject have not been placed in separate sections of the book, but are combined and intermingled as occasion requires It must be clearly understood, however, by the student that the formal course of theory is logically complete in itself, and he will find no difficulty in distinguishing this course from the remainder of the book, as it consists exclusively of definitions, axioms, and propositions, moreover the distinction will be further emphasised in a very practical manner, as he will be expected to commit the argument of the formal course to memory—though *not*, of course, to learn its words "by heart."

The authors of this book fear that at the present day, when so much stress is laid upon training of the intelligence, the value of memory-work is apt to be overlooked. It is still generally admitted that children have no fair chance of developing good literary taste unless they are made to learn by heart a certain amount of good poetry; for in no other way can their attention be riveted word by word on the text, and their ear be trained to the pulse and music of the verse. In the same way there is no better method of developing a scholar's logical faculty than to make him commit to memory models of clear and convincing argument, and to criticise thoroughly his attempts to reproduce or to imitate them.

We now proceed to describe some of the more important details in the plan of the book:—

Introductory Course.—This book commences with an Introductory Course of Drawing and Measurement designed to give the pupil a thorough and intelligent acquaintance with the elementary concepts of Plane and Solid Geometry as representing concrete and accurately measurable quantities. The details of this course are worked out very thoroughly, but its range is purposely restricted. It deals almost exclusively with plane and solid rectilineal figures; no mention is made of area or volume; and only a few of the simplest properties of plane rectilinear figures are demonstrated empirically. These restrictions have been observed in the belief that too much empirical knowledge is a positive disadvantage to young pupils commencing the study of geometrical theory, and that as soon as the nature of geometrical magnitudes is thoroughly understood and geometrical instruments can be handled with ease, the theoretical and practical studies should proceed side by side.

Logical Sequence of Propositions.—The order of proof adopted in this book follows closely the general plan which is indicated by the syllabuses of the Cambridge Local Examinations Syndicate for the Preliminary, Junior, and Senior Examinations. The authors believe that some standard order of logical sequence is much to be desired, and regard these syllabuses as the most authoritative pronouncement which has yet been made.

References.—The value of Euclid's concise system of references is well known by experience to all teachers who have ever used a Euclidian text-book, but this loss cannot be replaced until some standard order of proof is generally adopted. In the present work, however, the authors have introduced a system of references which it is hoped will be found of considerable value for ordinary class purposes. The system will be sufficiently explained by one or two examples; thus the abbreviation C.3 is used to indicate the third proposition on congruences, and the abbreviation T.2*c*. to indicate the converse of the second proposition on triangles. This plan has the advantage that the reference symbol to any proposition indicates to some extent the nature of the proposition and its position in the logical order of proof.

The authors have, however, been careful to arrange that *this system of references can be discarded if desired.* The proof of every proposition is complete without them —an obvious necessity—and an index is given at the end of the book from which the meaning of any reference can be found as easily as the numerical answer to any problem.

Hypothetical Constructions. — The usual hypothetical constructions are of course admitted and justified. The student is carefully warned against illogical hypothetical constructions—such as "Let ABCD be a square equal in area and perimeter to the triangle EFG"—a very necessary precaution in regard to the solving of riders.

Problems.—Geometrical Constructions do not form an essential part of the theoretical course. They are accordingly given in separate chapters at suitable intervals with proofs in small type; but the authors strongly advise that the proofs of Problems should be studied with as much care as the proofs of Theorems.

Exercises.—The book is well supplied with all types of exercises, viz. Riders, Drawing Problems, and Calculations. Problems involving diagrams drawn to scale are also given, but may be postponed if desired till the Theory of Similar Figures is reached.

The authors believe that the average boy will not acquire the faculty of solving riders without considerable help in the earlier stages. Accordingly numerous small sets of easy riders have been inserted. Also in the larger sets of more important riders, each of the first few riders is followed by a list of *all* the propositions required in the proof, and each of the next few by a reference to the leading propositions required. In the latter portion of the book these aids are partially withdrawn.

In the case of all numerical problems it is clearly indicated whether they are to be solved by drawing and measurement or by calculation. Wherever possible an accurate numerical result is required of the student.

Preliminary Exercises.—Heuristic methods are easily carried to excess, but used in moderation they are an essential element of good teaching; for the mind of the pupil should be kept active and alert, not passively recipient. Thus, when introducing a new proposition, the teacher should obtain the proof mainly from the class. In most cases this is not difficult, especially if any necessary construction is given; but when approaching the harder propositions, the ground should be cleared by means of a few suitable riders. Such riders are given in this book under the name of *Preliminary Exercises.* The figures of these riders, with perhaps a few notes, should be left on the board for the class to consult when the main proposition is attacked.

Range.—The present volume contains all that is required for the London University Matriculation Examination and for the Senior Oxford Local Examination: that is to say, it includes a thorough treatment of the subject matter of Euclid, Books I–IV.

A First Course of Reading.—It has been found necessary to introduce at various points certain matter which is too difficult for beginners. Accordingly in a first reading students should omit all articles, sets of exercises, or single exercises which are marked with an asterisk. The subject-matter of Chapter I is necessarily difficult It is essential on the one hand that the beginner should learn and understand the Definitions and Axioms, and on the other that he should not be troubled with the more difficult footnotes. With these restrictions the treatment of the Chapter must be left entirely to the discretion of the teacher; but it is well to point out that Exercises XVIII and XIX have been made as easy as possible and are intended mainly for discussion between the teacher and the class.

W. P. W.
A. G. C.

July 1906.

CONTENTS.

PAGE
INTRODUCTORY COURSE OF DRAWING . . 1
CHAPTER I.—THE SCIENCE OF GEOMETRY . 64
CHAPTER II.—ANGLES AND PARALLELS . . 77
CHAPTER III.—TRIANGLES . . 104
CHAPTER IV.—PARALLELOGRAMS . . . 136
CHAPTER V.—PROBLEMS . . . 147
CHAPTER VI.—LOCI 158
CHAPTER VII.—AREAS . . . 170
CHAPTER VIII.—PROBLEMS ON AREA . . 199
CHAPTER IX.—THE RIGHT-ANGLED TRIANGLE 209
CHAPTER X.—THE CIRCLE SYMMETRY . 220
CHAPTER XI.—THE CIRCLE CHORDS AND TANGENTS . 229
CHAPTER XII.—THE CIRCLE. ANGULAR PROPERTIES 251
CHAPTER XIII.—THE CIRCLE · PROBLEMS 277
CHAPTER XIV.—MISCELLANEOUS PROPOSITIONS 302
CHAPTER XV.—THE COMPOSITION OF RECTANGLES 330
CHAPTER XVI.—RECTANGLE-THEOREMS ON TRIANGLES AND CIRCLES . 346
CHAPTER XVII.—PROBLEMS ON RECTANGLES . 369
CHAPTER XVIII.—REGULAR POLYGONS . 381
List of Propositions 400
Answers 413
Index to Terms and Definitions . . 434

INTRODUCTORY COURSE

OF

DRAWING AND MEASUREMENT.

1 **Points.**—Figure 1 consists of four dots and one spot To distinguish between them, the dots are named by the letters *A*,*B*,*C*, and *D*, and each letter is written close to its own dot , also the spot is named by the letter *E*

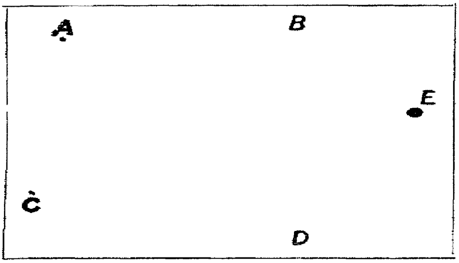

FIG 1

Each dot has a certain definite position on the figure , thus the dot *B* is 2 inch from the line at the top of the figure and 7 inch from the line on the right of the figure , while the dot *C* is 1 2 inches from the line at the top of the figure and 1 8 inches from the line on the right of the figure Also we can compare the positions of the dots one with another Thus the dot *A* is 1 inch from the dot *B* and 1 inch from the dot *C* , while the dot *D* is 1 25 inches from *B* and 1 6 inches from *A*

It is not easy to describe the position of the spot *E* exactly , for *E* is not a mere dot but is fairly large, and its different parts are in slightly different positions The top part of *E* is rather nearer to *B*, or to the line at the top of the figure, than the bottom part , and the left side of *E* is rather nearer to *C* than the right side of *E*

In reality the four dots A, B, C, D also have size, though they are much smaller than the spot E; thus the dot A has parts, and its different parts are not in *exactly* the same position. Hence to think of an *exact* position we must think of a dot *without any size at all.* Such a dot is called a **point**; we cannot draw a point, so we indicate its position as accurately as we can by as small a dot as possible.

DEFINITION.—A *point* has position, but no parts and no magnitude.

The point of a needle is not a true point in the geometrical sense; nor is the point of a fine hair, though this is much smaller than the needle point; nor even the point of a wasp's sting, which is very much smaller.

2. **Lines.**—If you draw the point of a pencil across a piece of paper, you will trace out a line. It may be a *straight* line or a *crooked* line; it will probably not be straight unless you use a "ruler" or "straight-edge" in the ordinary way.* Thus we may regard a **line** (whether straight or not) as **the path traced out by a moving point.**

If the point of your pencil is blunt you will get a broad or "thick" line, if it is sharp you will get a narrow or "thin" line; but if the point of your pencil were a true geometrical point you would get a line *without any breadth at all.* Thus in geometry we regard a true **line** as having **length but not breadth.**

DEFINITION.—A *line* is the path traced by a moving point; it has length but not breadth.

We can give several instances of very thin lines; *e. g.* the edge of a razor, and the thread which a spider spins to weave its web. But of course a geometrical line, like a geometrical point, does not exist, but is merely a convenient conception.

Most of the lines in the diagrams in this book are made fairly thick in order that they should be as clear as possible.

---

* In ruling a line do not work the pencil from the fingers or from the wrist, but as far as possible move the arm bodily from the elbow.

If you draw two points, A and B, on a piece of paper (see Fig. 2) you can always draw one (and only one) *straight* line from one point to the other; that is to say, you can **join** the points A and B by a straight line in one way only. This straight line is known as the **join** of the points.

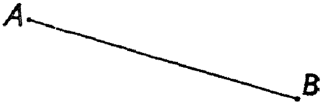

FIG. 2.

The definition of a **straight line** will be given later in the book.

The easiest way to *test* whether a line is straight, is to see whether a tightly stretched piece of fine cotton can be made to lie exactly along the line. Another method is to look along the line with the eye at the edge of the paper; if the line is straight, it will then appear as a point.

A third method is to trace the line (say the line in Fig. 2) carefully on tracing paper; then turn the tracing paper over and apply the trace to the original line, fitting the trace of A to the original point A, and the trace of B to B. If the line is straight, the trace will now fit accurately on to it at every point.

3. **Points on Lines.**—If you are told to put the point of your pencil on the line in Fig. 2, you can put it in many different positions; for instance, you can put it at the point A, or the point B, or anywhere between A and B. Each possible position of your pencil-point may be regarded as a **point on the line.** Thus the *extremities* of line are *points* (A and B in Fig. 2), and there are any number of other *points* on the line.

We may if we like roughly regard the line as *made up of* a very large number of points set very close together.

A line is usually named by means of the points at its extremities. Thus the extremities of the line in Fig. 2 are the points A and B, and the line should be called "the line AB." We may, however, name a line by means of any two points on it, whether they are at the extremities of the line or not.

4. **Intersections of lines.**—In Fig. 3 the two lines AB and CD cross one another; in this case there is one point, viz. O, which lies on both lines. We express this by saying that the two lines AB and CD **meet** at the point O; or that the two lines **cut** (or **intersect**) at the point O.

The point O is called the **point of intersection** of AB and CD, or the **cross** of AB and CD.

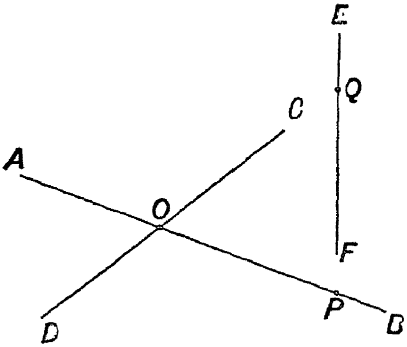

FIG. 3.

The straight line EF does not meet the straight line AB; but if the straight line EF were **produced through** F (that is, lengthened beyond F) it would meet AB at the point P. In the same way if DC were produced through C it would meet EF at the point Q.

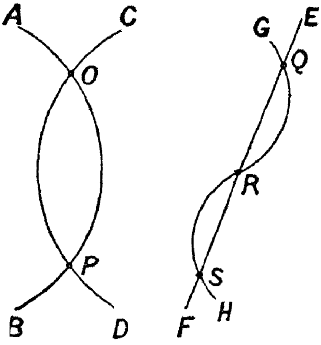

FIG. 4.

Two *straight* lines will never meet at more than one point, but if two lines are not both straight they may meet at any number of points. Thus in Fig. 4 the curved lines AB and CD meet at *two* points, viz. O and P; while the straight line EF and the curved line GH meet at *three* points, viz. Q, R, and S.

The method of representing points by dots is not satisfactory; for the dots must be made too large, or else they are not easily seen—especially if the points lie on lines as in Figs. 3 and 4. The best way to represent a point on a line is by a short thin stroke drawn across the line at the point in question; see the points *H* and *K* in Fig. 5. If the point is not on a line it is best indicated by a pair of short thin strokes intersecting at the point; see the point *L* in Fig. 5.

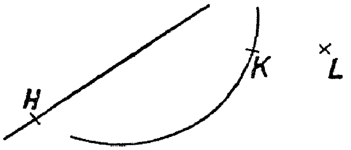

FIG. 5.

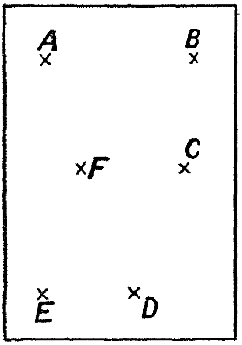

FIG. 6.

*EXERCISES I.**

Take a half-sheet of note-paper and mark six points on it, putting them roughly in the positions shown in Fig. 6. Join the following pairs of points by straight lines:—

1. *A* and *B*. **2.** *F* and *C*. **3.** *E* and *D*. **4.** *A* and *E*. **5.** *B* and *C*.
6. *A* and *C*. **7.** *B* and *D*. **8.** *E* and *F*.

9. At what point does the line *AB* meet the line *BC*?

10. At what point does the line *EF* meet the line *AE*?

11. Use the letter *P* to mark the cross of the lines *AC* and *BD*.

12. Use the letter *Q* to mark the cross of the lines *FC* and *BD*.

13. Produce the line *EF* through *F*; use the letter *R* to mark the point where *EF* produced meets *AC*.

14. Use the letter *S* to mark the point where *EF* produced cuts *AB*.

15. Use the letter *T* to mark the point where *BC* produced intersects *ED* produced.

16. If one vessel is steaming straight from *B* to *D*, and another straight from *F* to *C*, will there necessarily be a collision? If there is a collision, where will it occur?

---

* While the class is working through the Exercises in this Introductory Course, it will frequently be advisable for the teacher to help by drawings on the blackboard.

5. **Measurement of Lines.**—The ruler used to draw and measure lines should be at least 6 inches long. One edge should be marked with a scale of inches and tenth-inches, and the opposite edge with a scale of centimetres and millimetres; note that a millimetre is one-tenth of a centimetre.

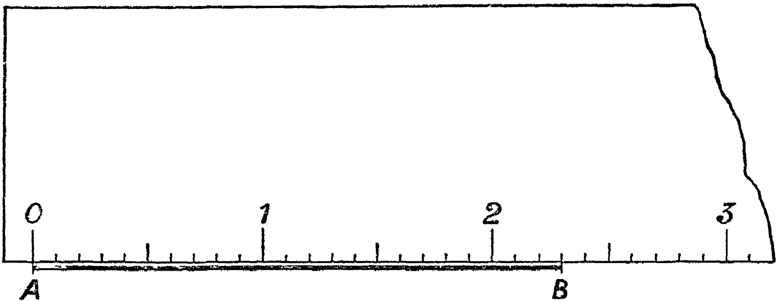

Fig. 7.

To measure the length of a line AB in inches (see Fig. 7) place the scale of inches so that the mark 0 on the scale falls at one end of the line (say at A) while the scale itself lies along the line. If the end B then falls exactly at one of the division-marks (or **graduations**) of the scale, you can read off the length at once; thus in Fig. 7 the length of the line AB is easily seen to be 2 inches and 3 tenth-inches. Using the ordinary decimal notation we express this as 2·3 inches, or 2·3″

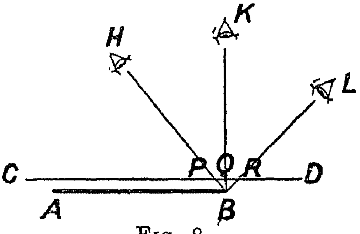

Fig. 8.

6. **Hints on using a scale for accurate measurement.** — The edges of the ruler (or protractor) which carry the scales for measuring inches and centimetres should be bevelled, so that the graduations are as near to the paper as possible.

While measuring the line the student must arrange that the edge of the ruler which he is using for the measurement does not cast a shadow on the paper; for if it does the line will be either in the shadow or else too far from the scale for accurate measurement. If possible the student should sit so that the light is coming from his left.

It is important for accurate work that the student's eye should be directly over each end of the line as he compares it with the scale; otherwise there is danger of a "**parallax error.**" The nature of this error is shown by Fig. 8, where AB represents the line and CD the scale used to measure the line. The scale CD is represented as above the paper on which the line is drawn; this is necessarily the case to some extent even though the scale is marked on a bevelled edge. If the eye is at H the point B appears to correspond to the point P on the scale; and if the eye is at L, B appears to correspond to R; Q is the point on the scale which really corresponds to B, and K is the correct position for the eye.

In reading from a scale only one eye should be used, as the difference in position of the two eyes causes some uncertainty as to the reading.

It is advisable to indicate the exact position of the end of the line by a short thin cross-stroke (see Fig. 9).

To get the scale as near to the paper as possible it is well to tilt the ruler so that only the edge which carries the scale is in contact with the paper.

EXERCISES II.

1. Measure each of the lines A, B, C (Fig. 9) in inches and tenths.

2. Measure each of the lines D, E, F (Fig. 9) in centimetres and tenths.

3. Measure each of the lines D, E, F (Fig. 9) in inches and tenth-inches, correct to the nearest tenth-inch. [That is to say, if the end of the line does not fall exactly at any graduation of the scale, read to that graduation which is *nearest the end of the line*, whether this graduation lies beyond the line or not.]

4. Measure each of the lines A, B, C (Fig. 9) in centimetres, correct to the nearest tenth-centimetre.

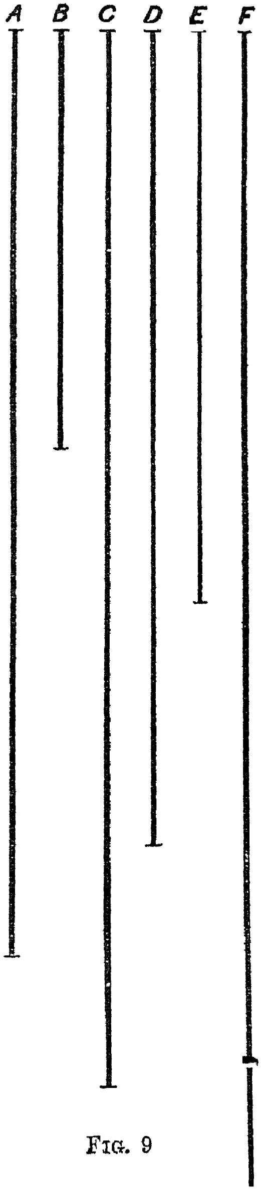

FIG. 9

5. Fig. 10 is a line divided into inches by the marks 0, 1, 2, 3. Four points, *A, B, C, D,* are marked on the line.

Now imagine that each inch is divided into tenth-inches, and guess * to the nearest tenth-inch the distance of each of the points *A, B C, D* from the point marked 0; test the accuracy of your results by measuring.

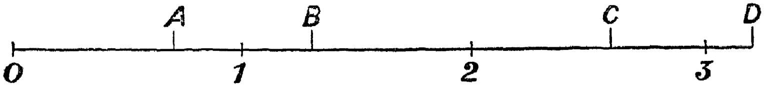

FIG. 10.

6. The line in Fig. 11 is divided into centimetres by the marks 0, 1, 2, 3, etc. Guess the distances of *A, B, C, D, E* from the point marked 0, correct to the nearest tenth-centimetre, and check your results by measuring.

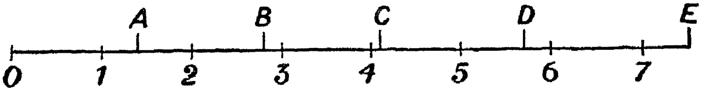

FIG. 11.

7. Draw two lines on paper each 4 or 5 inches long; use the scales to divide one into inches and the other into centimetres; mark points on these lines at random,† guess the distance of each point from the point marked 0, and test your results by measuring.

7. **Measurements correct to three figures.**—When using a scale to measure a line in the usual way, the end of the line does not as a rule fall exactly at one division of the scale, and the length of the line cannot be expressed exactly in inches and tenth-inches, or in centimetres and millimetres. Fig. 12 shows a line *CD* with the scale of inches placed to measure it. If the line ended at the point *S* instead of at *D*, its length would be 2 inches and 1 tenth-inch; but the extra length *SD* forms a fraction of the next tenth-inch on the scale. Now imagine this tenth-inch (which is marked *ef* in the diagram) to be divided into ten equal parts; each part will be one-tenth of a tenth-inch, that is *one hundredth of an inch.* Next *try to estimate how many of these equal parts* into which *ef* has been divided *will be covered by the line SD.* More than half of them certainly; that is more than 5;

---

* To aid in your guessing fix mentally the positions of the middle points of the inches, *i. e.* of the three lengths 01, 12, and 23.

† For this exercise the class may be divided into pairs, each boy making the random marks for his partner, not for himself. It will usually be found that a boy gets certain divisions more wrong than others; let him note these.

probably 7 is the correct number. Hence we estimate *SD* to be 7 hundredths of an inch.

Thus the whole length *CD* is 2 inches 1 tenth-inch and 7 hundredths of an inch; using the ordinary decimal notation we express this as 2·17 inches, or **2·17″**.

In the same way the scale of centimetres can be used to measure a line correct to tenths of a millimetre. If the result is 5 centimetres 6 millimetres and 3 tenths of a millimetre we should express this as **5·63 cms.**, or perhaps as **56·3 mms.**

The *distance between* two points is the length of the "join" of the points; but in order to measure this distance it is not necessary to draw the line.

The scales can also be used to mark off a required length along a line; in order to do this accurately a well-sharpened pencil is required.

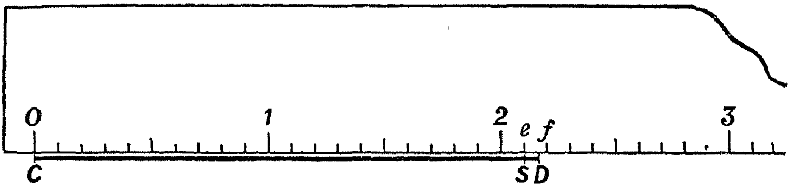

FIG. 12.

8. **On Pencil Points.**—The student should always have two pencils ready for use, the one sharpened to a fine "needle-point," and the other to a "chisel-edge" (*i. e.* to a flat edge like the edge of a chisel). In using a chisel-edge the pencil should be made to travel in the direction of its edge. A small piece of sand-paper is very useful to finish off a chisel-edge and to keep it sharp as it wears.

All *lines* on a drawing should be drawn with the chisel-edge, which will draw lines quite as thin as a needle-point, while it wears blunt much more slowly; all *points* on a drawing should be marked with the needle-point, as the chisel-point is not accurate enough for this purpose.

9. **The Compasses.**—Each compass-leg should measure about 3·5 or 4 inches, as this is the most convenient size for handling. The pencil should be sharpened to a chisel-edge, and fixed in the compasses so that the pencil-point just projects beyond the steel-point when the compasses are closed; the pencil should be set so that in using the compasses the pencil travels along the chisel-edge. The compasses should be held by the head only; the steel-point should be sharp, and should only just be allowed to prick the paper.

10. The Circle.—*To draw a circle with centre A and radius* 2·35″ (see Fig. 13).

Using the scale of inches mark two points *E* and *F* on the paper exactly 2·35″ apart; join them by a line if you like.

Adjust the compasses so that the distance between the steel-point and the pencil-point is exactly equal to the distance *EF*.

Place the steel-point at *A*, and turn the compasses round it with the pencil in contact with the paper; you will thus trace out the **circle** *BCD*.

Note that in order to name a circle we usually mention *three* points on it.

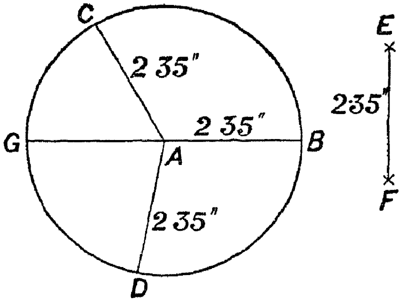

FIG. 13.

Strictly speaking the curved line which you have drawn is not the circle; it is the *space inside the curved line* which is called the circle.* The curved line is called the **circumference** of the circle, and any part of this circumference is called an **arc**. Thus *CG, CB, BD* are arcs.

The point *A* is called the **centre** of the circle.

Join *AB, AC* and *AD*. Any straight line which joins the centre of a circle to any point on its circumference is called a **radius** of the circle; thus *AB, AC, AD* are all *radii* of this circle.

---

* There are, however, many phrases in which this distinction is ignored, *e. g.* "a point on a circle," "the intersections of two circles," etc.

Now the line *AB* is equal to the distance between the steel-point and the pencil-point of the compasses; for the steel-point was kept at *A*, and the pencil-point passed through *B* while the circle was being traced. But the distance between the two points of the compass is equal to *EF*. Hence *AB* is equal to *EF*.

In the same way any other radius of the circle can be shown to be equal to *EF*, and is therefore 2·35″ in length.

All radii of the same circle are equal.

All points on the circumference of a circle are the same distance from the centre. Thus all points on the circumference of the circle you have drawn are 2·35″ from the centre *A*.

Any straight line through the centre of the circle terminated at both ends by the circumference is called a **diameter** of the circle. Thus, in Fig. 13, *GB* is a diameter.

Any diameter is made up of two radii. Thus the diameter *GB* is made up of the two radii *AG* and *AB*.

Since in any circle all radii are equal, it follows that all diameters are equal. Thus in the circle which you have just drawn each diameter measures 2 × 2·35″ or 4·7″.

Any diameter divides a circle into two halves, each of which is called a **semi-circle**.

We shall use the symbol ⊙ to denote a circle.

If a circle is drawn on paper and the paper be bent round, the figure is no longer a circle. The surface of a saucer is not a circle because it is not flat. A circle is a flat or plane figure.

DEFINITION.—A *circle* is a plane figure bounded by one curved line called its circumference, which is such that all points on this circumference are at equal distances from a certain point which is called the centre of the circle.

EXERCISES III.

In Fig. 14 measure* the following lines, (i) in inches, correct to the nearest hundredth of an inch, (ii) in centimetres, correct to the nearest hundredth of a centimetre :—

1. *AB*. 2. *CD*. 3. *EF*. 4. *GH*. 5. *KL*. 6. *MN*. 7. *MO*.

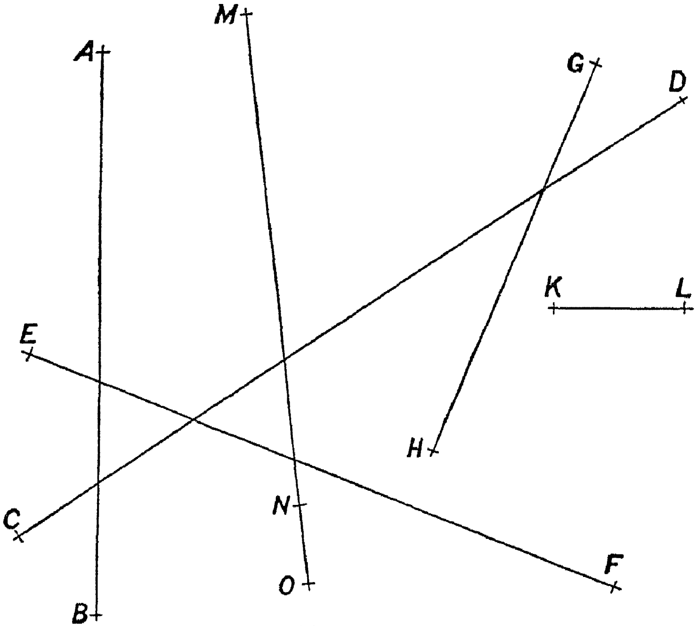

Fig. 14.

Also measure the distances between the following pairs of points, (i) in inches and decimals, (ii) in centimetres and decimals :—

8. *A*,*H*. 9. *N*,*L*. 10. *G*,*K*. 11. *A*,*F*. 12. *C*,*E*.

---

* The following is a good system for marking the answers to Exercises III. :—

For results in inches, give two marks if the answer is correct, or if the error is not more than ·02 inch; give one mark if the error is not more than ·04 inch. For results in centimetres, give two marks if the answer is correct, or if the error is not more than ·03 centimetre; give one mark if the error is not more than ·06 centimetre.

It should be remembered that an absolutely correct answer is largely a matter of chance.

Fig. 15 is a map drawn to a scale of 20 miles to the inch; that is to say, one inch on the map represents a distance of 20 miles on the country. Find the distances between the following pairs of towns:—

13. *R*, *T*. **14.** *R*, *S*. **15.** *P*, *T*. **16.** *P*, *Q*.

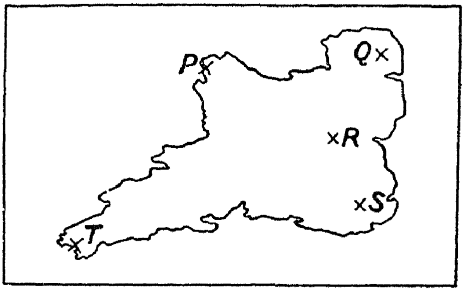

Fig. 15.

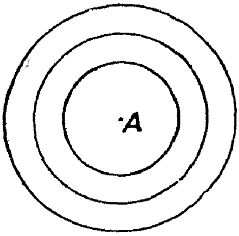

Fig. 16.

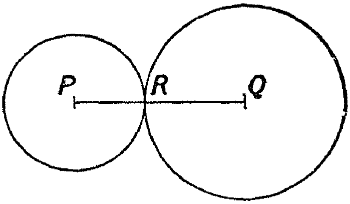

Fig. 17.

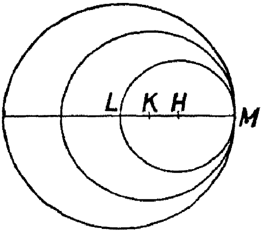

Fig. 18.

17. Draw three circles having the same centre *A*, as in Fig. 16.* The radius of the smallest circle is to be 1″, of the second circle 1·5″, and of the largest circle 2″.

18. Draw two circles with centres *P* and *Q*, as in Fig. 17. *PR* is to measure 1·6 cms., and *QR* 2·3 cms. [First draw the line *PQ*.]

19. Draw three circles with centres *H*, *K*, *L*, as in Fig. 18. *HM* is to be ·6″, *KM* ·9″, and *LM* 1·2″. [First draw the line *LM*.]

---

* A good way of correcting pupils' exercises in this and in more complicated cases, when a numerical measure is not easily available, is to have the exact figure drawn on tracing paper and apply it to the pupils' drawings.

11. The Triangle.—

DEFINITION.—A *triangle* is a plane figure bounded by three straight lines.

Four triangles of different shapes are shown in Fig. 19. The three lines which form the triangle are called its **sides**; thus in the triangle *DEF* the sides are *DE, EF, FD,* The side *EF* is sometimes called the **base** of the triangle, as it is the side upon which the triangle appears to be standing; but as we can suppose the triangle to be turned round into any position, we can suppose any side to be the base.

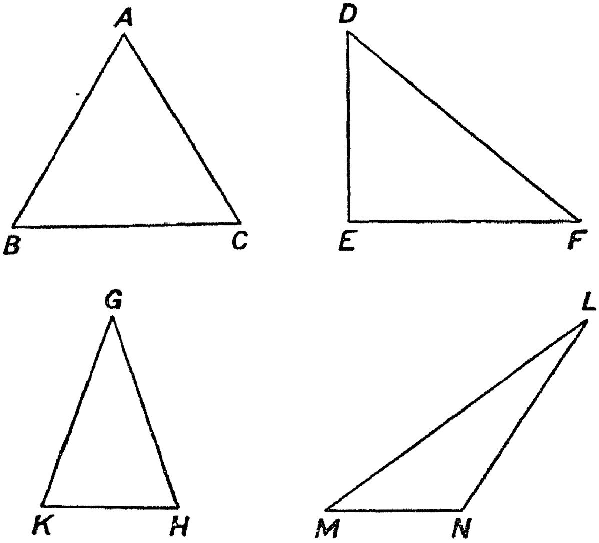

FIG. 19.

The three corners of the triangle are called its **angles**; hence the name "triangle," which means a three-angled figure. The exact meaning of the term angle will be explained later.

Each of the three points at the corners of a triangle is called a **vertex** of the triangle (plural *vertices*), or an **angular point**; thus the points *A,B,C* are the vertices of the triangle ABC, and the points L, M, N are the vertices of the triangle LMN. Distinguish between the angle and the angular point; the angle is the corner, while the angular point (or vertex) is the point at the "tip" of the corner.

If one side is called the base of a triangle the opposite angular point is called "*the vertex*" or sometimes the **apex** of the triangle; the angle opposite to the base is called the **vertical angle.** Thus in the triangle GHK, if we call HK the base we may call G the apex; or if we call GK the base we may call H *the* vertex.

We shall use the symbol Δ to denote a triangle.

If all three sides of a triangle are of equal length, it is called an **equilateral triangle.** Thus in Fig. 19 ABC is an equilateral triangle.

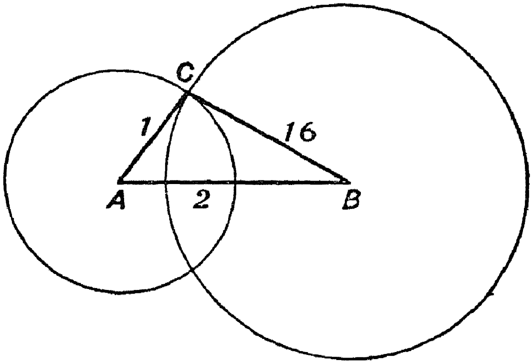

FIG. 20.

12. **Problem I.**—*To construct a triangle, given its three sides.*

Let us construct a triangle having its base 2 inches long, its left side 1 inch long, and its right side 1·6 inches long:—

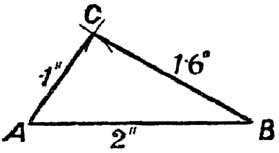

FIG. 21.

Draw the base AB of length 2 inches (Fig. 20). Describe a circle with centre A and radius 1 inch. Describe a circle with centre B and radius 1·6 inches. Take the point C above AB where the circles cut one another, and join it to A and B by straight lines. ABC will be the required triangle.

For the side AC is a radius of the left hand circle and is therefore 1 inch long; and the side BC is a radius of the right hand circle and is therefore 1·6 inches long.

It is not necessary to draw the complete circles, but only the parts of the circles (or "arcs") in the neighbourhood of the point C. See Fig. 21.

It will be found impossible to draw a triangle having one of its sides greater than the other two together. Try to draw a triangle whose sides measure 3, 1·5 and 1 inches; and another whose sides measure 20, 30, 60 mms.

Corollary.—To draw an equilateral triangle of side 1·5″ we should use the method given above; making AB 1·5″ long, and drawing each circle with a radius of 1·5″.

EXERCISES IV.

Describe triangles whose sides have the following lengths *:—

1. Base, 2″; left side, 2″; right side, 1″.
2. Base, 2″; left side, 1·5″; right side, 2·5″.
3. Base, 3″; left side, 2″; right side, 1·5″.
4. Base, 30 mms.; left side, 70 mms.; right side, 50 mms.
5. Each of the three sides to measure 27 mms.
6. Base, 2·6″; left side, 1″; right side, 2·4″.
7. Base, 30 mms.; left side, 39 mms.; right side, 60 mms.
8. Base, 8 cms.; left side, 5 cms.; right side, 5 cms.
9. Base, 3″; left side, 3·6″; right side, 2″.
10. Base, 3″; left side, 3·3″; right side, 2″.
11. Base, 2·5″; left side, 1″; right side, 1·2′.
12. Base, 1″; left side, 1·2″; right side, 3″.

13. **On angles.**—We have already mentioned that the corners of a triangle are called *angles*. If the student examines the triangles which he has just drawn and the triangles in Fig. 19, he will see that some of the corners are sharp, so that if the triangle were made of metal these corners could be used for piercing. These sharp corners are *small* angles, and the more blunt corners are *large* angles. Thus, in the triangle LMN in Fig. 19, the angle at N is large, the angle at M is small, and the angle at L is still smaller.

---

* Mark the lengths of the sides on the figures, as in Fig. 21.

An angle is not necessarily a part of a triangle. **Any two lines drawn from one point form an angle.** Thus in Fig. 22 four angles are drawn of which the angle *A* is the smallest, the angle *D* the largest, and the angle *C* is larger than the angle *B*.

When the two legs of a pair of compasses are opened they form an angle; and the *wider* we open them, the *larger* is the angle which they form. We could set the two legs of our compasses into each of the positions represented by the angles in Fig. 22.

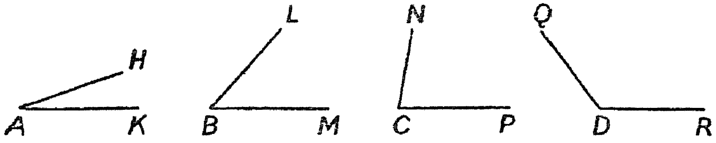

Fig. 22.

The two lines which form an angle are called the *arms* **of the angle, and the point where the two arms meet is called the** *vertex*, or sometimes the *angular point*. Thus in Fig. 22 the lines *AH* and *AK* are the two arms of the angle *A*, and the point *A* is the vertex of the angle or the angular point; in Fig. 19 the lines *LM* and *LN* are the arms of the angle *L*.

14. **On the magnitude of an angle.**—We must now define more carefully what we mean by the **size of an angle**; we can do this in two different ways, and both methods are important:—

(i) We may say that the size of an angle depends on the **difference in the directions of its arms.** Thus in Fig. 22 the lines *AK* and *AH* point in nearly the same direction, and hence the angle *A* is small; but *DR* and *DQ* point in very different directions, and hence the angle *D* is large.

(ii) In Fig. 22 if a line is placed in the position *AK* it could be turned round *A* into the direction *AH*, or if a line is placed in the position *DR* it could be turned round *D* into the position *DQ*; but *the amount of turning required in the second case would be much more than the* *amount required in the first.* Hence we may say that the magnitude of an angle depends on **the amount of turning** necessary to turn a line from the position of one arm of the angle to the position of the other arm.

It is very important to notice that the *size of an angle does not depend on the lengths of its arms* but merely on the shape at the corner. In Fig. 22 if we lengthen or shorten one or both of the arms of the angle A we should not alter the size of the angle, so long as we do not alter the shape at the corner A.

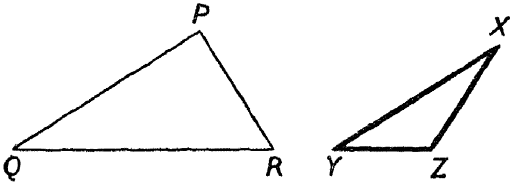

Fig. 23.

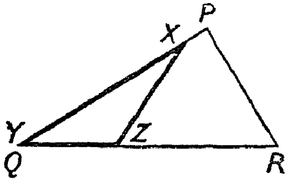

Fig. 24.

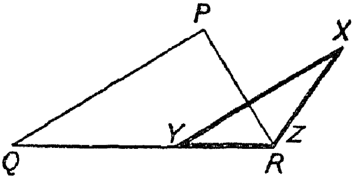

Fig. 25.

Two angles are equal if we can place one angle on the other in such a way that the corners fit. Thus the angle at Q in the triangle PQR (Fig. 23) is equal to the angle at Y in the triangle XYZ; for, if the triangle XYZ were cut out from the page, we could place the angle at Y on to the angle at Q in such a way that the corners at Y and Q would exactly fit (as shown in Fig. 24). But the angle at R is smaller than the angle at Z (Fig. 23), for if we placed the corner Z on to the corner R we should obtain Fig. 25, from which it is evident that the arms RQ and RP are less wide apart than the arms ZY and ZX.*

---

* Instead of cutting out △ XYZ trace it on a piece of tracing-paper, and then place the tracing over △ PQR.

Up to the present we have been naming an angle by a letter placed at the angular point. This method is satisfactory so long as there is only one angle at that point; but in other cases it may lead to confusion. The correct way to name the first angle in Fig. 22 is "the angle HAK"; notice the order of the letters; in the *middle* we place the angular point A, while the letters H and K are points one on each arm of the angle. It does not matter whether we place H or K first or last so long as we keep A in the middle; so that we may call the angle "KAH" if we choose. Similarly the fourth angle in Fig. 22 may be called either QDR or RDQ. In Fig. 21, the angle at the left-hand end of the base should be called the angle BAC (or CAB), and the angle at the top of the triangle should be called the angle ACB (or BCA).

We shall use the symbol $\angle$ for angle.

This symbol is very convenient in print; but in writing it is liable to be mistaken for the letter L. In order to avoid this the student should turn the symbol round and place it, as a circumflex accent, over the letter which represents the angle, or over the middle letter if three letters are used.

EXERCISES V.

Turn to the triangles in Fig. 19 :—

1. Name the angles at C, E, G and N.

2. Name the angle at the right-hand end of each base.

Turn to the triangles in Fig. 23 :—

3. What are the arms of the angle whose vertex is P?

4. What are the arms of the angle whose vertex is Y?

5. Name the angle whose arms are QP and QR.

6. Name the angle whose arms are ZY and YX.

7. Which do you consider the smallest angle in the triangle PQR?

8. Which do you consider the largest angle in the triangle PQR?

9. Which do you consider the largest angle in the triangle XYZ?

10. Is the angle at P greater or less than the angle at X?

11. Test your answers to questions 7–10 by the use of tracing-paper.

Turn to Fig. 19 and answer the following questions by the use of tracing-paper :—

12. Of the three angles B, E, H which is the greatest and which the smallest?

13. Of the three angles D, F, M which is the greatest and which the smallest?

14. Are any two angles of the triangle GHK equal?

15. Are any two angles of the triangle ABC equal?

15. **On the composition of angles.**—Now consider Fig. 26. We have eleven different lines drawn from *P*, and they form ten small equal angles, viz. the angles *APB, BPC, CPD, DPE,* etc. The arms of these angles are of different lengths, but the angles themselves have been drawn equal.

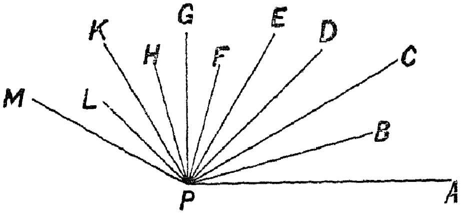

FIG. 26.

Any two of the lines which meet at *P* form an angle; for example the lines *PC* and *PE* form an angle *CPE.* Also it is obvious that this angle *CPE* is made up of two of the small equal angles, viz. *CPD* and *DPE.* Thus this angle *CPE* is twice as large as the angle *CPD* or twice as large as the angle *APB.*

In the same way the angle *EPH* is made up of three of the small equal angles,—viz. *EPF, FPG,* and *GPH.* Hence the angle *EPH* is three times as large as the angle *APB.*

EXERCISES VI.

Using Fig. 26 :—

1. How many times is ∠*APB* contained in each of the following angles ?—(*a*) *DPF,* (*b*) *FPL,* (*c*) *APG,* (*d*) *MPA.*

2. Name the line which divides ∠*KPE* into two equal parts.

3. Name the line which divides ∠*KPC* into two equal parts.

4. *PM* is one arm of an angle which is five times as great as ∠*APB* ; which is the other arm ?

5. How many times is ∠*BPD* contained in ∠*LPF* ?

6. How many times is ∠*MPH* contained in ∠*GPA* ?

7. How many times is ∠ *GPE* contained in ∠*MPA* ?

8. Name the two angles into which the line *PD* divides the angle *BPG.*

9. What angle is left when ∠*APB* is taken away from ∠*APD* ?

10. What single angle is made up by the two angles *BPD* and *DPG* taken together ?

16. **The Right Angle.**—Take a piece of paper *AHBKL* (Fig. 27); the shape does not matter so long as *AB* is a straight line.

Take any point *H* on *AB*,—near to *B* by preference; fold the paper in such a way that the point *A* falls on to the point *H*, as shown in Fig. 28; the crease *CD* will be found to be a straight line. Then the angle *DCB* or any angle equal to *DCB* is called a **right angle**; and we can tell at once whether any angle is a right angle by trying to fit this folded piece of paper on to it.

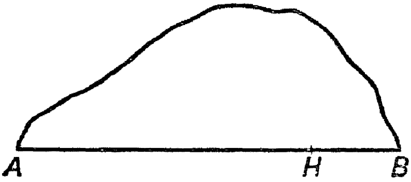

FIG. 27.

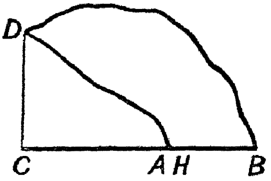

FIG. 28.

Right angles are very common, and are easily recognized. The angles at the corners of an ordinary sheet of note-paper, or at the corners of the cover of a book are all right angles; also the angles of an ordinary picture-frame, box-lid, black-board, etc.

Your two "set squares" are wooden triangles, and in each of these one angle is obviously a right angle. Any triangle which has one angle a right angle is called a *right-angled triangle.*

If two lines form a right angle they are said to be **at right angles** to each other, or **perpendicular** to each other. Thus the side edge of a black-board is perpendicular to the bottom edge, or to the top edge.

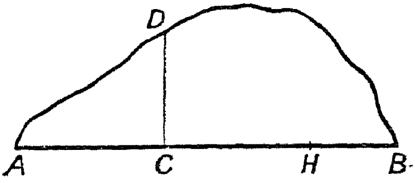

FIG. 29.

Now unfold the piece of paper and spread it flat, marking the crease *CD* as in Fig. 29. Then the angle *DCA* is equal to the angle *DCB*, for it fitted on to it when the paper was folded; hence $\angle DCA$ is a right angle.

Thus DC is perpendicular to both CA and CB; that is, DC is perpendicular to the whole line AB.

We shall use the symbol $\perp$ for perpendicular.

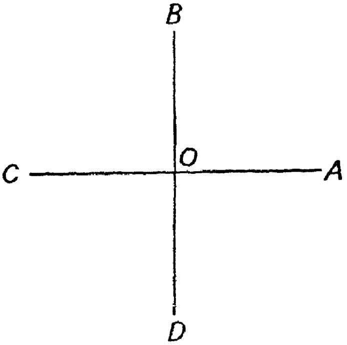

FIG. 30.

In Fig. 30 each of the angles AOB, BOC, COD, DOA is a right angle. Now suppose a line to start from the position OA and, turning round O as a pivot, to pass through the positions OB, OC, OD, and to come back to OA; this line is said to have made a *complete revolution.* But in doing this the line has traced out each of these four right angles in succession; hence we obtain the following

DEFINITION.—A *right angle* is the angle described by a line in turning through one-quarter of a complete revolution.

Fig. 30 can be obtained by folding any piece of paper *twice.* We may suppose Fig. 27 to represent the paper after the first folding, the straight line AB being the crease. Fig. 28 will then represent the paper after the second folding. If the paper is now unfolded and spread flat, the creases will form four right angles, as in Fig. 30.

Angles which are less than right angles are called **acute** (from a Latin word meaning *sharp*), and angles which are greater than right angles are called **obtuse** (from a Latin word meaning *blunt*). Thus in Fig. 22 all the angles are acute except the last.

17. The following problems must now be practised till the student can do them from memory:—

Problem II.—*At a given point C in a given straight line AB, to draw a line perpendicular to AB.*

We must first draw attention to a common but bad method of using a set square to draw a perpendicular. Take either of the set squares and place it in the position represented by the triangle HKL in Fig. 31, that is, with the right-angled corner K at C, and with one edge KL lying along CB. Holding it in this position with the left hand, rule a line along the edge HK with the right hand. Then HK is ⊥ to LK, and therefore to AB.

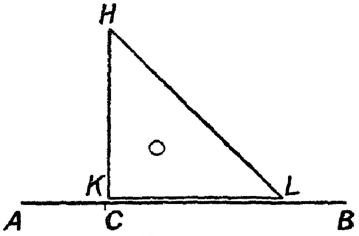

FIG. 31.

This method should never be adopted, for it is difficult to rule the line HK quite as far as the line AB, and accurately to the point C. Moreover, after a little use, the corner of the set square is apt to get slightly rounded and inexact.

FIRST METHOD.—Take one of the set squares and place it in the position represented by the dotted triangle HKL (Fig. 32); that is, so that one arm KL of the right angle lies along AB and *projects beyond* C (say by a quarter or a half of an inch). Now place the second set square against the first in the position MNO, so that one edge MO lies along the edge LH. Holding MNO firmly in place, slide the other set square along it into the position PQR, so that the edge PQ can be used to rule a line through C. Holding PQR firmly, rule the line PC, which will be the required perpendicular.

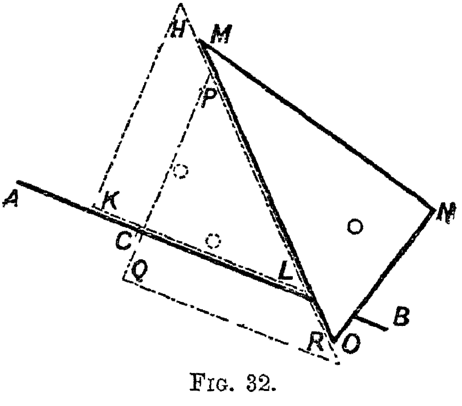

FIG. 32.

Practise this method several times and take the point C in different positions on the line.

NOTE 1.—If the perpendicular is to be drawn on both sides of the line AB, the first position HKL of the set square should be farther to the left than is here represented.

NOTE 2.—It is advisable that neither the edge KL, which is adjusted to AB, nor the edge PQ, which is used to rule the perpendicular, should be allowed to cast a shadow; but this is much more important for KL than for PQ. If the light is coming from the left the arrangement of the set squares sketched in the figure avoids a shadow in both cases; the paper should be turned so that AB slopes down to the right, as here represented.

NOTE 3.—The straight edge may be used instead of the set square OMN.

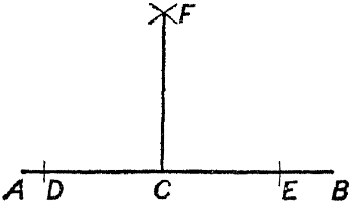

FIG. 33.

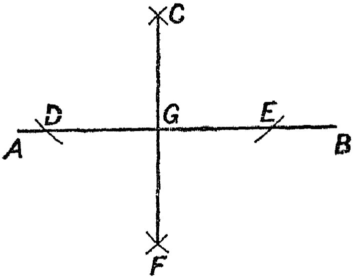

FIG. 34.

SECOND METHOD.—By setting the compasses to a convenient radius and placing the steel-point at C, mark off equal distances CD, CE on each side of C (Fig. 33). With centres D and E, and any convenient equal radii, describe arcs intersecting at F.[*] Join CF.

Then CF is the required perpendicular.

NOTE 1.—If the point C is near to the end of the line it will be necessary to produce the line (*i. e.* lengthen it by means of the ruler) before we begin the construction given above.

NOTE 2.—Try the effect of using a shorter radius at D than at E.

---

[*] When a point is determined by the intersection of two arcs (as the point F in this construction) the exact position of the point is most easily determined if the two arcs cut (roughly) at right angles. To obtain a good intersection in this figure, the equal radii should be taken at about three-quarters of DE.

Problem III.—*From a given point C without a given straight line AB, to draw a line perpendicular to AB.*

First Method.—Take one of the set squares and place it in the position represented by the dotted triangle HKL (Fig. 35). Place the second set square against the first in the position MNO. Now hold MNO firmly in its place and slide the other set square along it into the position PQR, so that the edge PQ can be used to rule a line through C. Holding PQR in this position rule the line PQ, which will be the required perpendicular.

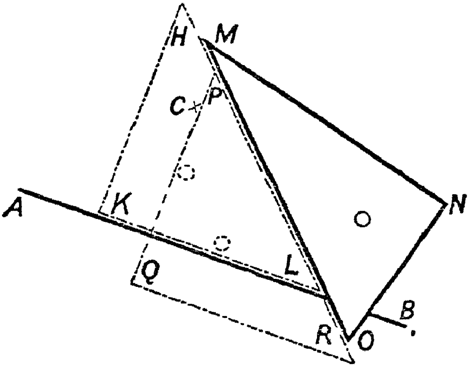

Fig. 35.

Note.—The straight edge may be used instead of the set square OMN.

Second Method.—With centre C and any convenient radius draw arcs cutting AB in D and E (Fig. 34). With centres D and E and any convenient equal radii * describe arcs cutting one another in F (on the other side of the line from C). Join CF cutting AB at G. Then CG is perpendicular to AB.

Note 1.—Here again it may be necessary to produce AB, before the above construction is possible.

Note 2.—The point G is called the *foot* of the perpendicular CG.

---

* About three-quarters of DE.

Problem IV.—*Given a line, AB, to* **bisect** *it, or divide it into two equal parts.* (See Fig. 36.)

With centres *A* and *B* and any convenient equal radii (about three-quarters of *AB*) describe two arcs cutting at *C* above the line, and two others cutting at *D* below the line. Join *CD*, cutting *AB* in *P*. Then *P* will bisect *AB*.

NOTE 1.—*CD* will always be perpendicular to *AB*.
NOTE 2.—Never bisect a line by the divisions on your ruler.
NOTE 3.—Try the effect of using a shorter radius at *A* than at *B*.

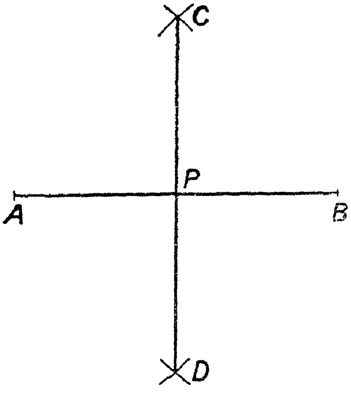

FIG. 36.

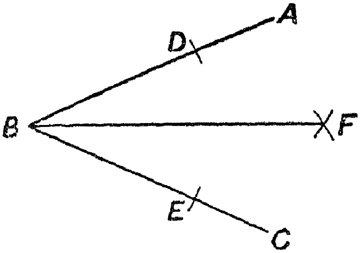

FIG. 37.

Problem V.—*To bisect a given angle ABC.* (See Fig. 37.)

Using the compasses mark off from *BA, BC*, any equal lengths *BD, BE*. With centres *DE*, and any convenient equal radii (about three-quarters of *DE*) draw two arcs cutting at *F*. *BF* will bisect the angle *ABC*.

NOTE.—Never use the protractor (Fig. 38) to bisect an angle.

EXERCISES VII.

Draw again the first ten triangles of Exercises IV. :—

1. In the first triangle bisect the base, and measure the distance of its middle point from the vertex.

2. In the second triangle bisect the vertical angle by a line cutting the base, and measure the two parts into which the base is divided.

3. In the third triangle draw a line from the vertex $\perp$ to the base and measure the part of the base which lies to the left of this line.

4. In the fourth triangle draw a line ⊥ to the base from its right-hand end to meet the left side; measure the length of this perpendicular.

5. In the fifth triangle draw a line at right angles to the base from its left extremity to meet the right side of the triangle produced; measure this perpendicular.

6. In the sixth triangle bisect the left angle by a line cutting the right side, and measure the parts into which the right side is divided.

7. In the seventh triangle draw a perpendicular from the vertex to the base produced, and measure the distance between the foot of this perpendicular and the nearer end of the base.

8. In the eighth triangle draw a line from the vertex perpendicular to the base, and measure the parts into which the base is now divided.

9. In the ninth triangle draw a line perpendicular to the base from its middle point, to meet the left side. Measure this line and the two parts into which the left side is divided.

10. In the tenth triangle bisect the base and the right side, and measure the length of the line joining the two points of bisection.

18. **On the measurement of angles.**—For the purpose of measuring angles, we divide a right angle into 90 equal parts, and we call each part a **degree.** A degree is obviously a very small angle.

In order to state the size of an angle, we state how many degrees it contains; obviously half a right angle will contain 45 degrees, two-fifths of a right angle will contain $\frac{2}{5}$ of 90 or 36 degrees, and so on. We use the symbol $^\circ$ to denote degrees, so that 23° means an angle containing 23 degrees.

In Fig. 26, GPA is a right angle, and it is divided into six equal parts. Hence each of these parts contains 15 degrees. It follows that

$$\angle BPD = 30^\circ,\ \angle DPG = 45^\circ, \text{ etc.}$$

One of your set squares has the arms containing the right angle of equal length; each of the acute angles in this set square measures half a right angle, or 45°. In the other set square the larger of the two acute angles contains two-thirds of a right angle, or 60°, and the smaller acute angle contains one-third of a right angle, or 30°. You should not, however, use the set squares for drawing angles of 30°, 45°, or 60°.

The number of degrees in an angle can be measured by means of a **protractor** (Fig. 38). In this instrument the centre *O* of the edge *AD* may be called the "pole" of the protractor, and is clearly marked by a line;

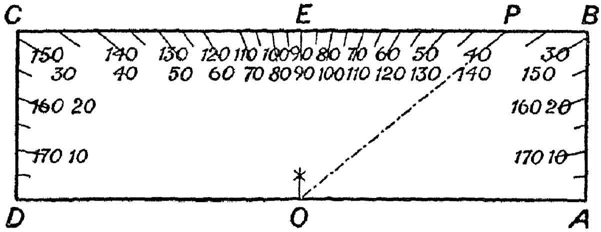

FIG. 38.

the other three edges *AB*, *BC*, *CD* carry a series of division-marks or "graduations," which are numbered from 0 to 180—the mark 0 occurs at *A*, the mark 90 at *E* the centre of *BC*, and the mark 180 at *D*.

It is a great advantage if these graduations are *also* numbered from 0 at *D* to 180 at *A*, as shown by the inner set of numbers in Fig. 38.

For the sake of clearness Fig. 38 shows only the graduations corresponding to every five degrees; but almost all protractors carry intermediate graduations showing degrees, and in some half-degrees are also shown.

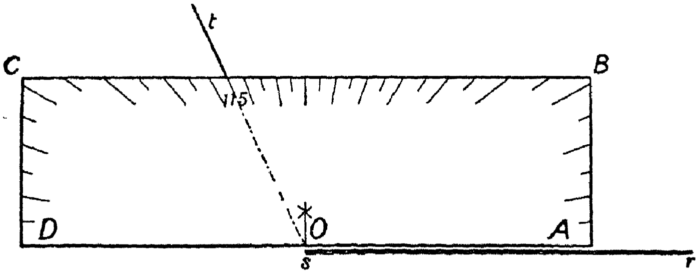

FIG. 39.

Fig. 39 shows the method of measuring an angle *rst*. The protractor is placed in the position represented in the diagram, so that the pole *O* lies exactly at the angular point *s*, and the edge *OA* of the protractor lies along the arm *sr* of the angle; the number of degrees in the angle is then given by that **graduation** of the protractor which falls on the other arm *st*. Thus in Fig. 39, the protractor shows that the magnitude of the angle is 115°.

If we wish to measure the same angle, using the system of graduations whose zero is at *D*, we should place the protractor so that the pole *O* falls exactly at the angular point *s*, and the edge *OD* of the protractor lies along the arm *st* of the angle. The number of degrees in the angle is then given by that graduation which falls on *sr*.

Explanation.—In Fig. 38 if we suppose *E* joined to *O*, the angles *EOA*, *EOD* are both right angles; if we also suppose that each of the division marks along the edges* is joined to *O*, then each of these right angles will be divided into 90 equal parts, that is, into degrees. Hence the outer number marked on the protractor at any point *P* shows the number of degrees in the angle *AOP*, or in any angle on to which the angle *AOP* will exactly fit.

NOTE 1.—The division-marks along the edges are not placed at equal distances, but are so placed that the angles formed by joining them to *O* will be equal.

NOTE 2.—If the second arm of the angle is too short to project beyond the protractor, it must be produced.

In Fig. 40 we have a line *OH* "standing upon" a line *KOL*; these lines then form two angles, viz. *HOL* and *HOK*. If the protractor *ABCD* is placed in the position shown by the dotted lines, it is easily seen that these two angles *HOL*, *HOK* together make up the total number of degrees shown on the protractor, *i. e.* 180° or two right angles. Hence *if one straight line stands upon another straight line the two angles so formed are together equal to two right angles.*

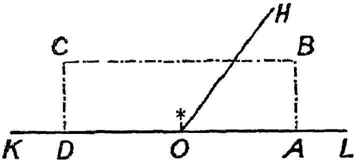

FIG. 40.

DEFINITION.—If two angles are together equal to two right angles (i. e. 180°) they are said to be *supplementary*.

---

* Including the intermediate graduations not shown on the diagram.

19. **The Circular Protractor.**—Protractors of the kind described in the last article (usually made of wood) are probably the most suitable for ordinary purposes. Another common form of protractor is semicircular, and is made of metal (see Fig. 41). The graduations in this case occur at equal distances along the rim; they are usually harder to read and more trying to the eyes than the graduations on a wooden protractor. In a metal protractor the position of the pole is often indicated by a notch; the *top* of the notch should then be taken as the pole.

Circular protractors can also be obtained in celluloid; these have the advantage of being transparent.

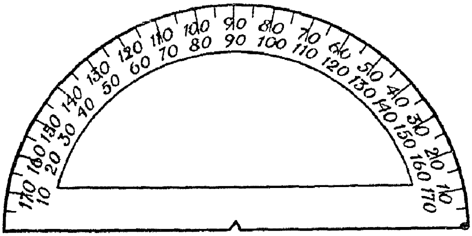

FIG. 41.

20. **On the accurate measurement of angles.**—In measuring angles the student should try to estimate correctly to tenths of a degree. The results may not be as reliable as in the measurement of lines; but it is well to set a high standard of work. Moreover, in practical work fractions of any kind should be expressed in decimals.

Theoretically in measuring an angle we start from the zero graduation; but in practice it will be found better with almost any protractor to avoid the zero graduation, as it is very seldom correctly placed with regard to the other graduations.

Thus in measuring the angle *rst* (Fig. 39) we can set the pole *O* to the angular point *s*, and set (say) the graduation 40 to the arm *sr*; it will then be found that the arm *st* coincides with the graduation 155, and hence that the magnitude of the angle is 155 − 40 degrees, *i.e.* 115°.

21. On Testing Instruments.

The Straight Edge.—To test the accuracy of a ruler, use it *twice* to rule a line through the same two points; the distance between the two points should be rather less than the length of the ruler. In the second ruling the *same edge* must be used as in the first, but the ruler must be turned over and must lie on the opposite side of the points. If the second ruling of the line coincides exactly with the first the edge is accurately straight.

Compare this test with the third test for a straight line given in § 2.

The Set Square.—To test the right angle of a set square, use the instrument *twice* to rule a line perpendicular to a given line *AB* at a given point *C*. In the first ruling arrange the work as in Fig. 32, where *PQR* is the angle of the set square to be tested. In the second ruling arrange the work as in Fig. 42. If the two rulings exactly coincide the right angle is accurate.

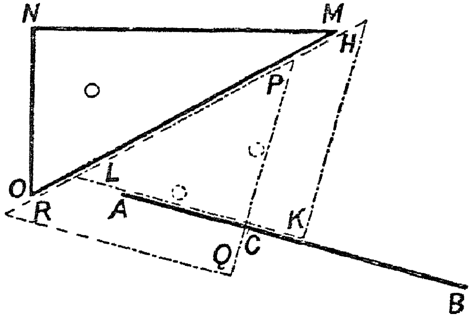

FIG. 42.

Explanation.—In the first case the line ruled makes, *with the line CB*, an angle equal to that of the set square. In the second case the line ruled makes, *with the line CA*, an angle equal to that of the set square. If therefore the angle of the set square is an accurate right angle each of these ruled lines will coincide with the true perpendicular; but if not the two ruled lines will lie one on each side of the true perpendicular.

The Protractor.—To test a protractor draw accurately an equilateral triangle of convenient size (say each side about 3·5 inches).

Measure one angle with the protractor, (i) using the zero graduation of the instrument, (ii) using the graduations 10°, 20°, 30°, etc., successively instead of the zero graduation (see § 20).

In each case the result should be 60°.

The Compasses.—The possible faults in the compasses are (i) a bad steel point, (ii) a loose hinge, (iii) a loose grip of the pencil.

EXERCISES VII.

Questions 1—10. Draw again the triangles of Exercises IV.; measure the three angles in each triangle, and mark at each angle the number of degrees which it contains.* From the results of these measurements try to answer the following questions :—

11. What is the total number of degrees in the three angles of any triangle?

12. If you know the magnitudes of the three sides of a triangle, can you tell which of its angles will be the greatest and which the smallest?

13. If a triangle has two of its sides equal, can you make any statement as to its angles?

14. If a triangle has all three sides equal, can you make any statement as to its angles?

When you know the correct answers to these four questions commit them to memory.

15. Draw any two lines cutting one another; there will be four angles formed at the point of intersection; measure each of these angles. Draw two other pairs of intersecting lines in different positions, and measure each of the angles in each case. From your results try to discover some laws with regard to the four angles formed by any two intersecting lines.

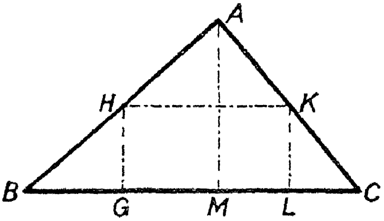

FIG. 43.

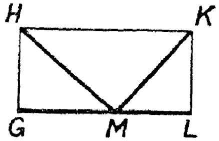

FIG. 44.

16. Draw any triangle† ABC (Fig. 43); if the triangle contains an obtuse angle or a right angle, mark this with the letter A; draw $AM \perp$ to BC. Now cut the triangle out (with scissors or a sharp pen-knife), and fold the corners over in such a way that each of the angular points falls on the point M. You will then find that the three

---

* A good system of marking for this exercise will be to give full marks if the error is not greater than ·4 degree, and half marks if the error is not greater than ·8 degree.

† For convenience in working make the longest side of the triangle about four inches long, and do not make any of the angles very small.

"flaps" so formed (*i. e.* the three parts which are folded over) will between them exactly cover the remainder of the triangle. [The lines *GH*, *HK*, *KL* in Fig. 43 show the creases made in folding; and Fig. 44 shows the appearance of the triangle when folded.]

What does this prove with regard to the three angles of the original triangle, which are now folded over to the point *M*?

17. Draw any four-sided figure. Measure its angles. What is their sum?

What do you infer about the angles of four-sided figures?

22. **On the construction of angles.**—To construct an angle (say of 135°) given one arm *PQ*, and the angular point *P*;—place the protractor so that *O* falls at *P* and *OA* lies along *PQ* (Fig. 45). Then make a pencil mark *R* on the paper against the mark 135 on the protractor, remove the protractor, and join *PR*. *RPQ* is the required angle.

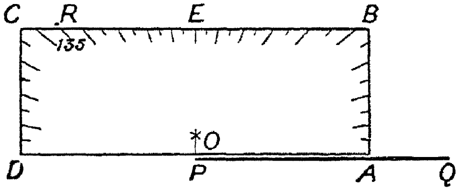

FIG. 45.

NOTE 1.—The above construction assumes that we are using that system of graduations whose zero is at *A*. If we wished to make an angle having *PQ* for one arm and *Q* for the angular point we should set the pole *O* to the point *Q*, and the edge *OD* to the line *QP*, and we should then use the system of graduations whose zero is at *D*.

NOTE 2.—For accurate work it is better, as we have already stated, to avoid the zero graduations.

NOTE 3.—The protractor may be used to rule a perpendicular to a given line; set *O* and *E* (Fig. 45) on to the given line and rule the perpendicular along *DA*. The objection to this method is that it assumes the accuracy of the zero graduation.

NOTE 4.—Certain angles can be constructed accurately without using a protractor. Thus by Problem II. we can construct an angle of 90° at the point *C* (Fig. 33). If in the same figure we bisect the angle *FCB* by a line *CG*, then $\angle GCB = 45°$, $\angle GCA = 135°$. By bisecting an angle of 45° we can obtain an angle of $22\frac{1}{2}°$, and so on.

Again we can construct an equilateral triangle by Problem I., and we then obtain an angle of 60° (see Exercises VIII., Ques. 14). Bisecting an angle of 60° we obtain an angle of 30°, and so on.

23. On the construction of triangles.—

Problem VI.—*To construct a triangle given two sides and the included angle.*

Let us construct a triangle having an angle of 72° and the sides containing that angle of lengths 2 and 3 inches respectively:—

Construct $\angle HAE = 72°$ (Fig. 46). From AE mark off $AB = 2''$; from AH mark off $AC = 3''$; join BC. ABC is the required triangle.

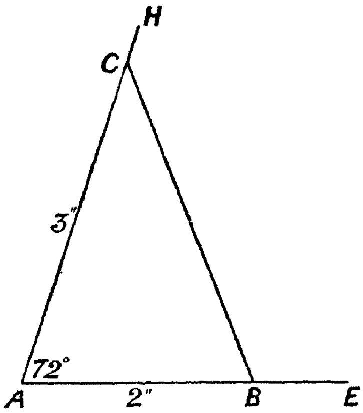

FIG. 46.

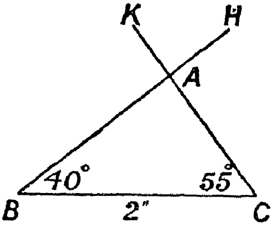

FIG. 47.

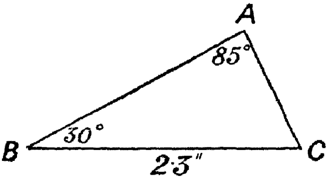

FIG. 48.

Problem VII.—*To construct a triangle given two angles and the side between them.*

Let us construct a triangle whose base is 2 inches, having angles of 40° and 55° respectively at the extremities of the base:—

Draw BC of length 2 inches (Fig. 47). Using the protractor, make an angle of 40° at B (viz. CBH), and an angle of 55° at C (viz. BCK).

If BH and CK intersect at A, ABC is the required triangle.

Problem VIII.—*To construct a triangle given two angles and the side opposite to one of them.*

Let us construct a triangle having two angles of 30° and 85°, and a side of 2·3 inches opposite to the second angle :—

It will be convenient first to make a rough sketch of the triangle (as in Fig. 48), and mark on the sketch the known magnitudes of the two angles and the side. It is now obvious that if we knew the number of degrees in the angle C we could use the construction of Problem VII.

But the three angles A,B,C are together equal to 180° (see Exercises VIII., Quest. 11 and 16) ; hence to find C we add 30° to 85° and subtract the result from 180° ; thus C is 65°. We can now construct a △ on a base 2·3 inches, having angles of 30° and 65° at the extremities of the base. For this use the method of Problem VII.

Problem IX.—*To construct a triangle given two sides and the angle opposite to one of them.*

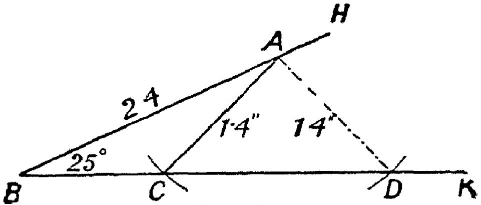

FIG. 49.

Let us construct a triangle having two sides of lengths 1·4 and 2·4 inches, and an angle of 25° opposite to the shorter side :—

Construct $\angle HBK = 25°$ (Fig. 49). From BH mark off $BA = 2·4$ inches.

Then A will be the vertex of the triangle, and B one end of the base. Also C, the other end of the base, lies somewhere along BK. We must now find where.

Since AC is the side opposite to the angle 25°, C must be 1·4 inches from A. But all points which are at this distance from A lie on a circle whose centre is A and radius 1·4 inches. Draw this circle.

C lies on the circle and also on the line; it must therefore be one of the points where the circle meets the line. But there are two such points, viz. those marked *C* and *D* in the figure.

Thus either of the triangles *ABC* or *ABD* may be given as the triangle required.

NOTE 1.—Notice the use of the circle (or arc) to draw the side *AC* (or *AD*). Whenever we wish to draw a line of *given length* from a *given point,* we may draw a circle with the given point as centre and the given length as radius. The other extremity of the required line must lie somewhere on the circumference of this circle. Compare the use of the circles in Problem I.

NOTE 2.—The three sides and three angles are sometimes called the six elements of a triangle. We cannot construct a triangle unless we are given three elements, including at least one side.

NOTE 3.—In constructing a triangle from *any* data it is a good practice first to sketch the triangle roughly and mark the given magnitudes, as in Problem VIII. It is then easy to see how to construct it accurately with instruments.

EXERCISES IX.†

1. With a base *AB*, 2″ long, draw the triangle *ABC*, whose angles at *A* and *B* respectively contain 60° and 30°. Measure the sides *AC* and *BC* and the angle *ACB*.

2. With base *AB* (3″) draw the triangle *ABC* whose angles at *A* and *B* contain respectively 45° and 75°. Measure the sides *AC* and *BC* and the angle at *C*.

Construct triangles *ABC* from the following data, and in each case measure the sides *BA, CA* and the angle *A* :—

3. $BC=1{\cdot}5''$, $B=40°$, $C=65°$. **4.** $BC=2''$, $B=50°$, $C=30°$.

5. $BC=2{\cdot}5$ cms., $B=60°$, $C=60°$. **6.** $BC=3$ cms., $B=72°$, $C=48°$.

Construct triangles from the following data, and measure the sides and angles not given :—

7. $AC=3$ cms., $AB=3$ cms., $A=90°$.

8. $BC=4$ cms., $A=90°$, $C=30°$.

9. $BC=3$ cms , $A=120°$, $C=15°$. **10.** $BC=3''$, $AC=2''$, $A=70°$.

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† The system of marking indicated in previous footnotes may be continued throughout the book. Another good system is to give full marks when the error is not greater than 1 per cent., and half-marks when it is not greater than 2 per cent.; but this method must never be pressed where it involves a greater accuracy than ·01 inch, ·02 cm., or ·2 degree. Marks should also be assigned for neatness and should not exceed one-quarter of the highest marks assigned for accuracy.

11. $AC=3$ cms., $AB=3$ cms., $A=36°$.

12. $AC=4$ cms., $AB=6$ cms., $C=120°$.

13. $BC=2''$, $AC=2{\cdot}5''$, $A=35°$.

14. $AC=6{\cdot}6$ cms., $AB=4{\cdot}4$ cms., $C=40°$.

In the next four examples M is the middle point of the side BC.

15. $BC=3$ cms., $B=50°$, $A=90°$; measure MA.

16. $BC=3{\cdot}5$ cms., $MA=2$ cms., $\angle AMC=120°$; measure AB.

17. $BC=4$ cms., $AB=2{\cdot}5$ cms., $MA=2{\cdot}5$ cms.; find AC and C.

18. $BC=4$ cms., $AB=2$ cms., $MA=2$ cms.; find AC, A and C.

19. Draw a line AB of length 2″; draw $AC \perp$ to AB. Use the compasses to find a point D on AC whose distance from B is 3″. Measure AD.

20. Draw a line AB of length 4 cms. On what curved line do all the points lie which are 2·5 cms. from A? On what curved line do all the points lie which are 2·5 cms. from B? Find two points, C and D, each of which is 2·5 cms. from A and 2·5 cms. from B. Measure the distance CD.

21. Draw a line PQ of length 2·5″. Find two points, R and S, each of which is 2″ from P and 1·5″ from Q. Measure the distance RS.

22. Draw a line HK of length 1·4″; draw $HL \perp$ to HK, and on HL find a point M whose distance from the point K is 2″. Next find a point N, which lies on the opposite side of MK to the point H, and which is 2″ from each of the points M and K. Measure the distance NH.

In the following exercises the protractor is not to be used.

Construct triangles from the following data :—

23. $BC=2{\cdot}2''$, $\angle B=45°$, $\angle C=45°$; measure AB and AC.

24. $BC=4$ cms., $\angle B=60°$, $\angle C=45°$; measure AB.

25. $AC=3''$, $BC=1{\cdot}5''$, $\angle C=60°$; measure AB.

26. $AB=2''$ $BC=2{\cdot}7''$, $\angle B=30°$; measure AC.

27. $BC=2''$, $\angle A=75°$, $\angle C=45°$; measure AB.

28. $AB=2{\cdot}8$ cms. $AC=4$ cms., $\angle A=105°$; measure BC.

24. On the construction of quadrilaterals.—

DEFINITION.—A figure bounded by four straight lines is called a *quadrilateral.*

For example $ABCD$ (Fig. 50).

The lines AC and BD which join opposite corners are called the **diagonals.**

It will be found that to construct a quadrilateral we must know five elements † (using the word "elements" to include sides, angles, diagonals, etc.). The problem can be set in many different ways, but at present the student may be content with the simpler cases.

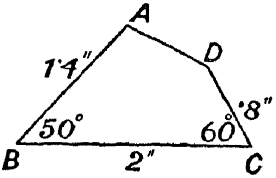


FIG. 50.

Problem X.—*To construct a quadrilateral given three sides and the two included angles.*

Let us construct a quadrilateral $ABCD$, such that $AB=$ 1·4 ins., $BC=2$ ins., $CD=\cdot 8$ ins., $B=50°$, $C=60°$:—

[If we first make a rough sketch of the quadrilateral we can see how to build it up from the given data. See Fig. 50.]

Draw $BC=2''$; make $\angle CBA=50°$ and arm $BA=1\cdot 4''$; make $\angle BCD=60°$, and arm $CD=\cdot 8''$. Join AD.

Problem XI.—*To construct a quadrilateral given the four sides and one diagonal.*

Let us construct a quadrilateral $ABCD$ such that $AB=$ 2 ins., $BC=2\cdot 5$ ins., $CD=3$ ins., $DA=3\cdot 5$ ins., $AC=$ 2·5 ins. :—

[If we first make a rough sketch as in Fig. 51 we see that we can construct $\triangle$s ABC and ADC, as we know the three sides of each. Compare Problem I.]

---

† The four angles of the quadrilateral must only count as *three* elements.

Draw $AC = 2{\cdot}5''$, with centres A and C and radii $2''$ and $2{\cdot}5''$ respectively describe arcs intersecting at B; with centres A and C and radii $3{\cdot}5''$ and $3''$ respectively describe arcs intersecting at D. $ABCD$ is the required quadrilateral.

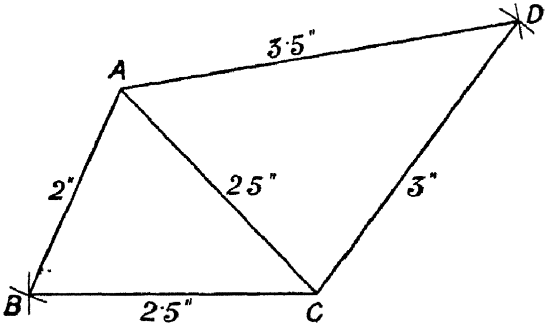


FIG. 51.

Problem XII.—*To construct a quadrilateral given the four sides and one angle.*

Let us construct a quadrilateral $ABCD$ given $AB = 1$ in., $BC = 1{\cdot}5$ ins., $CD = 1{\cdot}2$ ins., $DA = 1{\cdot}2$ ins., $B = 75^\circ$:—

[See Fig. 52. Imagine AC joined; we can construct $\triangle ABC$ by the method of Problem VI., as we know two sides and the included angle; we can then construct $\triangle ADC$ by the method of Problem I.]

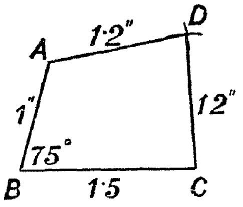


FIG. 52.

Draw $\angle ABC = 75^\circ$, and mark off on its arms lengths $BA = 1''$ and $BC = 1{\cdot}5''$. With centres A and C and radii $1{\cdot}2''$ describe arcs intersecting at D.

The student will be able to solve many other simple cases by first drawing a rough sketch, and finding from it how the figure can be built up from the given data.

EXERCISES X.

Construct quadrilaterals $ABCD$ from the following data :—

1. $AB = BC = 1''$, $CD = \cdot 3''$, $\mathrm{B} = 90^\circ$, $C = 90^\circ$; measure AD.

2. $AB = 1\cdot 5''$, $BC = 1''$, $CD = 1''$, $DA = 2''$, $AC = 2''$; measure BD.

3. $AB = 5$ cms., $BC = 3$ cms., $CD = 6$ cms., $DA = 1$ cm., $ABC = 90^\circ$; measure $\angle A$.

4. $DA = 2$ cms., $AB = 4$ cms., $BC = 4$ cms., $A = 60^\circ$, $B = 120^\circ$; measure $\angle C$.

5. $AB = 3$ cms., $BC = 3\cdot 5$ cms., $CD = 2\cdot 5$ cms., $DA = 3$ cms., $BD = 4\cdot 5$ cms.; measure $\angle B$.

6. $DA = 2\cdot 1''$, $AB = 1\cdot 4''$, $BC = 1''$, $CD = 1''$, $A = 50^\circ$; measure AC.

7. $AB = 2''$, $BC = 1\cdot 5''$, $CD = 2\cdot 2''$, $AC = 2\cdot 2''$, $BD = 2\cdot 2''$; measure AD.

8. $BC = 40$ mms., $ABC = 80^\circ$, $DBC = 35^\circ$, $DCB = 70^\circ$, $ACB = 40^\circ$; measure AD.

9. $BD = 32$ mms., $DA = 32$ mms., $BC = 20$ mms., $ABD = 30^\circ$, $DBC = 25^\circ$; measure AC.

10. $AB = 2''$, $BC = 2''$, $CD = 1''$, $ABC = 108^\circ$, $BCD = 108^\circ$; measure AD and $\angle A$.

11. $AB = 45$ mms., $BC = 39$ mms., $CD = 25$ mms., $AC = 42$ mms., $BD = 56$ mms.; measure AD.

12. $AB = 1\cdot 5''$, $BD = 2''$, $DC = 2\cdot 5''$, $ABD = 90^\circ$, $BDC = 90^\circ$; measure AC and $\angle CAB$.

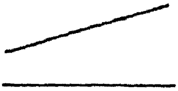


FIG. 53.

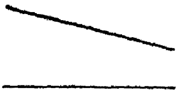


FIG. 54.

FIG. 55.

25. **On parallels.**—In Fig. 53 we have two lines which do not meet, but which would meet if they were both produced far enough to the left. We say that these lines *converge* (or run together) on the left, and *diverge* (or run apart) on the right. In Fig. 54 we have two lines which *diverge* on the left, and *converge* on the right.

In Fig. 55 we have two lines which do not meet however far we produce them in either direction, not even if we produce them any distance beyond the page. Such lines are called **parallel** lines.

It is not correct to say that *any* two lines are parallel which would not meet if produced; for example, if a line is drawn on the floor pointing north and south, and another is drawn on the table pointing east and west, these two lines would not meet if produced, but we should not call them parallel. The full definition of parallel lines will be given later; for the present it will be sufficient to say that any two lines *drawn on the same flat sheet of paper* are parallel if they do not meet when produced to any length in either direction.

Parallel lines are very common and are easily recognized. The lines on an ordinary sheet of ruled paper form a set of parallel lines, as no two of them meet or would meet even when produced beyond the paper. The top edge of a door is parallel to the bottom edge, and the left edge to the right edge.

Parallel lines point in the same direction. Thus, if you turn your book round till the lower line in Fig. 55 points north and south, the upper line in that figure will also point north and south.

We shall use the symbol || for parallel.

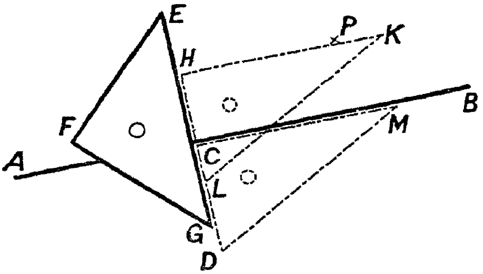


FIG. 56.

Problem XIII.—*To draw a line through a given point parallel to a given line.*

Let us draw through *P* a line || to *AB* (Fig. 56):—

FIRST METHOD.—Place one set-square so that one side lies along *AB*, as in the position *MCD* (Fig. 56). Place the other set-square against it, in the position *EFG*. Keeping *EFG* fixed, slide the first set-square along it into the position *KHL*, where the side *HK* is in position to rule a line through the point *P*. Rule the line *HPK*, which will be the required parallel.

NOTE 1.—The edge CM must not be allowed to cast a shadow.

NOTE 2.—Always make the sliding set-square slide along the *longest side* of the fixed set-square.

NOTE 3.—The arrangement of the set-squares shown in the figure is usually the most convenient; but in some cases it is more convenient to apply the set-square EFG to the side DM of the set-square DCM, instead of to the side CD; or to reverse DCM, placing the side DC on the right, and to apply EFG to the side DM.

NOTE 4.—The straight-edge may be used instead of the set-square EFG.

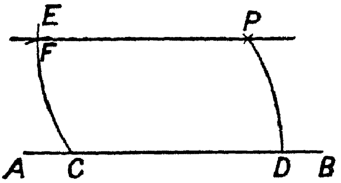


FIG. 57.

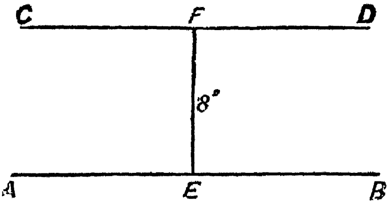


FIG. 58.

SECOND METHOD.—Take any convenient point C in AB (Fig. 57). With centre C and radius CP draw the arc PD, cutting AB in D. With centre P and radius PC draw the arc CE. From CE cut off $CF = DP$ (measuring with the compasses). Join FP, which will be the required parallel.

Problem XIV.—*To draw a line parallel to a given line at a given distance.*

Let us draw a line $\parallel$ to AB at a distance of ·8″ :—

FIRST METHOD.—At any point E in AB (Fig. 58) draw $EF \perp$ to AB, and of length ·8″ (Problem II.). Through F draw CD $\parallel$ to AB (Problem XIII.). CD will be the required parallel.

Every point in the line CD will be at a distance ·8″ from the line AB. Note that *the distance of a point from a line is measured by the length of the perpendicular drawn from the point to the line or to the line produced.* Thus in Fig. 58 the distance of the point F from the line AB is measured by the perpendicular FE; because F is nearer to E than to any other point in AB.

Again, if in Fig. 58, we are asked to find the *distance of the point C from the line BE*, we should measure the perpendicular drawn from C to BE produced; as *it is always assumed* that we may produce the line in question.

SECOND METHOD.—Take any two points in the line, H and K (Fig. 59). With centres H and K and radius ·8″ describe two arcs on the same side of the line. Draw CD *touching* each of these arcs (as shown in the figure). CD will be the required parallel.

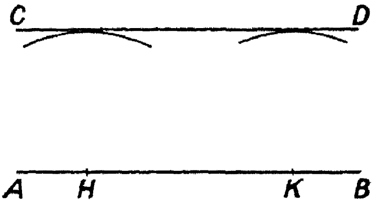


FIG. 59.

The second method is more rapid, but is less accurate in the hands of beginners.

EXERCISES XI.

1. Take a line AB about 2·5″ long. Bisect it at C. Draw lines $\perp$ to AB at the points A, C, and B respectively. Note that these lines are parallel. Join any point H on the first perpendicular to any point K on the third, and let HK cut the second perpendicular in L. Measure HL, LK. What do you infer?

2. Take a line AB of length 2·4″. Bisect it at C. Construct an angle $BAD = 40°$. Through C draw a line $CE \parallel$ to AD. From B draw a line $\perp$ to AD meeting AD in F and CE in G. Measure the angles BCE and BGE, and the lengths BG, GF.

3. Draw three parallel lines. Draw a line cutting all three lines and making an angle of 50° with one of them. Show that every angle in the figure measures either 50° or 130°.

4. Draw a line, a second parallel to it, and 1 inch above; a third parallel to the second, and 2 inches above. Draw a line making an angle of 42° with the lowest line, and cutting the other lines. Measure all the angles in the figure.

What do you notice with regard to any two angles in this figure?—(i) if both are acute, (ii) if both are obtuse, (iii) if one is acute and the other obtuse.

Commit these facts to memory.

5. Draw a line, and two other lines parallel to it. Draw a line at right angles to one of the lines and cutting the other lines, and measure all the angles in the figure.

What general rule is indicated by this example?

6. Draw AB of length 1·5″. Make an angle $BAC = 35°$, and an angle $ABD = 35°$ (AC and BD being on opposite sides of AD). Note that AC is $\parallel$ to BD. Make $AC = BD = 1{\cdot}7''$. Join CD, cutting AB in O. Measure AO, BO, CO, DO. Note that AB and CD bisect one another.

7. Construct an equilateral triangle of side 4 cms. Through the vertex draw a line ∥ to the base. Measure the distance from the base of any point on this parallel.

8. Draw a △ of sides 3, 4, 5 cms. Draw a line ∥ to the longest side through the opposite vertex. Measure the distance between the parallels.

9. Draw △ *ABC*, whose sides *AB*, *BC*, *CA* are 1·8″, 2·4″, 3″ respectively. Measure the distance of *B* from the opposite side.

10. Draw a triangle whose sides are 3, 5, 7 cms. respectively. Measure the distance of each vertex from the opposite side.

11. Draw an equilateral triangle of side 2·5″. Take any point within the triangle and find the sum of its distances from the three sides.

12. Draw a line *AB* about 2″ in length; in *AB* take any point *C*. Draw another line *DE* such that every point on *DE* is at a distance of 1·2″ from the line *AB* (or *AB* produced). Find two points, *F* and *G*, on *DE* each of which is 1·5″ from *C*. Measure *FG*.

13. Draw △ *ABC*; *AB* = 3 cms., *BC* = 4 cms., *CA* = 5 cms. Draw a line *EF*, between *A* and *BC*, such that every point on *EF* is 2 cms. from *BC*. Find a point *G* on *AC* which is 2 cms. from *BC*. Measure *CG*.

14. Draw a line *AB*, and in it take two points *C* and *D* one inch apart. Find two points *E* and *F* each of which is 1″ from the line *AB*, and 1·4″ from the point *C*. Measure the distances of *E* and *F* from *D*.

15. Draw an angle *BAC* of 58°. Find a point *D* on *AB* which is 1″ distant from the line *AC*. Measure *AD*.

16. Draw an angle *HKL* of 56°. Find a point *O* within the angle which is ·8″ distant from each of the arms of the angle. Measure *OK*.

17. Draw an equilateral triangle *ABC* of side 2″. Find a point *D* within the triangle which is at a distance of ·6″ from the sides *AB* and *AC*. Measure the distance of *D* from *A* and from the line *BC*.

18. Through a given point *O* draw two lines *AB* and *CD*, making the angle *AOC* equal to 60°. Draw the two lines on which all the points lie whose distance from *AB* is ·6″. Draw the two lines on which all the points lie whose distance from *CD* is ·6″. Find four points each of which is ·6″ distant from *AB* and ·6″ distant from *CD*. Measure the distance of each of these four points from *O*.

19. Draw two lines *AB*, *CD* intersecting at *O* at an angle of 70°. Find four points each of which is ·8″ distant from *AB* and *CD*. Measure the distance of each of the four points from *O*.

20. Draw an angle *ABC* of 75°. Find a point *K* within the angle whose distance from *AB* is 1″, and from *BC* is 1·4″. Measure *BK*.

21. Draw a triangle *ABC* having *AB* = 4·5 cms., *BC* = 6 cms., *CA* = 7·5 cms. Find a point within the △ whose distance from *AB* is 2 cms. and whose distance from *BC* is 1·2 cms. Measure its distance from *AC*.

26. On Parallelograms.—

DEFINITION.—A ***parallelogram*** **is a quadrilateral in which both pairs of opposite sides are parallel** (see Fig. 60).

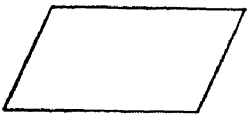


FIG. 60.

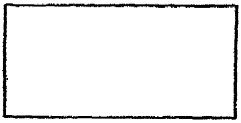


FIG. 61.

DEFINITION.—A ***rectangle*** **is a quadrilateral in which each of the four angles is a right angle** (see Fig. 61).

The rectangle is a very common shape; a sheet of note-paper, the cover of a book, a door—all of these are in the shape of rectangles.

In a rectangle both pairs of opposite sides will be found to be parallel; hence any rectangle is a parallelogram.

DEFINITION.—A ***rhombus*** **is a quadrilateral in which all four sides are equal** (see Fig. 62).

In a rhombus both pairs of opposite sides will be found to be parallel; hence any rhombus is a parallelogram.

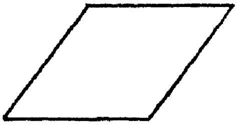


FIG. 62.

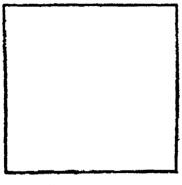


FIG. 63.

DEFINITION.—A ***square*** **is a quadrilateral in which all the four sides are equal and any one angle is a right angle** (see Fig. 63).

It will be found that all the angles of a square are right angles.

A square may be called a rectangle, a rhombus, or a parallelogram; as it possesses the distinctive characteristics of each type of figure.

EXERCISES XII.

1. Draw any two pairs of parallel lines such as a, b, and c, d in Fig. 64; produce them till each line in the one pair intersects each line in the other pair. You will thus form a parallelogram such as *PQRS*.

Use this method to draw four different parallelograms. Examine them, take any rough measurements you think desirable, and see whether you can discover any properties of parallelograms with regard to (i) the sides, (ii) the angles, (iii) the diagonals.

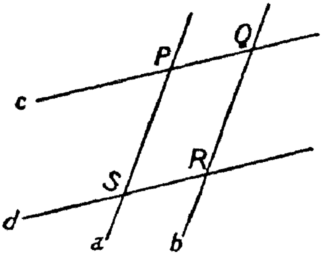


FIG. 64.

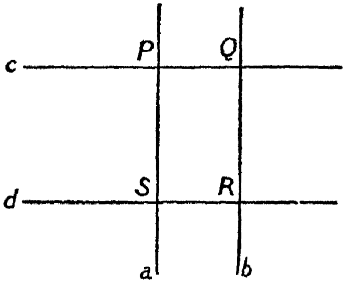


FIG. 65.

2. Draw any two parallel lines such as a, b (Fig. 65) and draw any two lines at right angles to one of them (and therefore also at right angles to the other) such as c, d. You will thus form a rectangle such as *PQRS*.

Use this method to draw four different rectangles. Examine them, take any rough measurements you think desirable, and see whether you can discover any properties of rectangles with regard to (i) the sides, (ii) the diagonals.

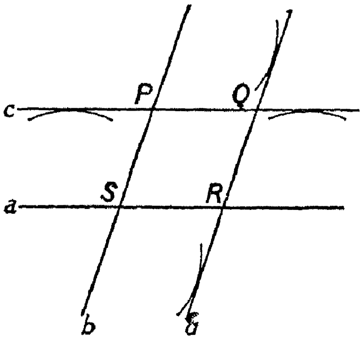


FIG. 66.

3. Draw any two intersecting lines such as a, b (Fig. 66); using Problem XIV., draw two lines c, d parallel to a, b respectively and at equal distances from a, b. You will thus form a figure *PQRS*. What kind of figure is it?

Use this method to draw four such figures all different; examine them, take any rough measurements you think desirable, and try to discover any of their properties with regard to (i) the angles, (ii) the diagonals.

4. If you use the construction of Question 3, *taking the lines a, b at right angles,* what kind of figure do you obtain?

Use this method to draw four such figures of different sizes; taking any rough measurements you think desirable, try to discover any properties with regard to their diagonals.

27. **Problem XV.**—*To construct a parallelogram given two intersecting sides and the included angle.*

Let us construct a parallelogram in which two sides measure 1″ and 2″ and the included angle measures 50° (see Fig. 67):—

Draw *AB* = 2″; make ∠*BAC* = 50°; make *AC* = 1″. Through *B* draw a line ‖ to *AC*, and through *C* draw a line ‖ to *AB*; if these lines meet at *D*, *ABDC* will be the required parallelogram.

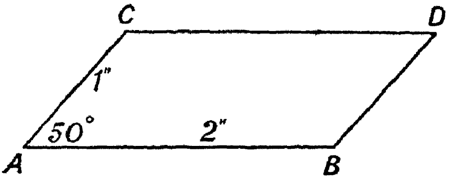


Fig. 67.

A similar construction enables us to describe a rectangle, square, or rhombus, as in all these figures opposite sides are parallel.

In the case of a rectangle or a square the angle at *A* (Fig. 67) must be made equal to 90°; and in the case of a square or a rhombus the sides *AB* and *AC* must be made equal.

EXERCISES XIII.

1. Draw a rectangle whose sides measure 3 and 4 cms., and measure both diagonals, and the parts into which these divide one another. What do you infer?

2. Draw a rhombus each of whose sides is 2″ and whose acute angle is 60°. Measure the diagonals, the parts into which they divide one another, and the angles at which they intersect.

3. Draw a parallelogram on a base 2″, having the left angle = 50° and the adjacent side = 1″. Measure the other three angles.

4. On base 1·5″ describe above it an isosceles triangle, having each side 2″, and another isosceles triangle below it having each side 2·5″, and measure the line joining the vertices of the two triangles. Measure the parts into which this line divides the base. What do you infer?

5. Draw a square of side 2″; measure both diagonals, the angles which they form at their intersection, and the parts into which they divide one another. What do you infer?

6. Draw any two straight lines *AB* and *CD* which bisect one another at a point *O*, at any angle. Join *AC*, *CB*, *BD*, and *DA*. Find the sum of the angles *DAC*, *ACB*; also of the angles *DAC*, *ADB*.

7. In the last example if *AB* were 6 cms. and *CD* 4 cms. and the angle *COA* 45°, measure all the other lines and angles.

8. On *AB* 2″ long as diagonal draw a parallelogram *ACBD* whose side *BC* is 1·5″ long and whose side *AC* is 2·5″, and measure the angles of the parallelogram and those at the intersection of the diagonals.

9. Construct the quadrilateral *ABCD* whose sides are 1″, 1″, 1·5″, and 1″, and one of whose diagonals is 1·7″. Measure the other diagonal.

10. Construct the quadrilateral *ABCD*. *AB* (1″) parallel to *DC* (2″) at a distance of 1″ from one another, the angle *BAD* being 120°; measure *AC*.

11. Construct a quadrilateral *ABCD*. *AB* (1·5″) parallel to *CD* (2″). *AD* being 2″, and the diagonal *AC* 3″. Measure the angles.

12. Construct a rectangle *ABCD*, given diagonal $AC = 2''$, and $\angle ACD = 40°$. Measure its sides.

13. Draw two lines of lengths 1·4″ and 2·4″ bisecting each other at right angles. Measure the sides of the quadrilateral of which these lines are diagonals.

14. Draw a circle of radius 1·5″. Draw two diameters cutting at an angle of 40°. Measure the angles of the quadrilateral of which these are the diagonals.

15. Draw a circle of radius 1·4″. Draw two diameters at right angles. Measure the sides and angles of the quadrilateral of which these are the diagonals.

16. Describe a rectangle in which a diagonal is 2·4″, and one side is 1·2″. Measure the longer side. (Start with the given side.)

17. Knowing that the diagonals of a parallelogram bisect each other, construct a parallelogram whose longer side is 3·5 cms., and whose diagonals are 5 and 4 cms. respectively. Measure the other sides.

18. Knowing that the diagonals of a square are equal and bisect each other at right angles, construct a square of diagonal 4 cms. Measure its sides.

Copy figures 68–73, making the lines and angles of the dimensions indicated on the figures. Also make the following measurements from your figures:—

19. In your copy of Fig. 68 measure *CE*, *CA*, and the angles *ABC*, *EAB*.

20. In your copy of Fig. 69 measure the lengths *AD*, *EB*.

21. In your copy of Fig. 70 measure *AB*, *BC*, and $\angle ABC$.

22. In your copy of Fig. 71 measure *AB* and $\angle AOC$.

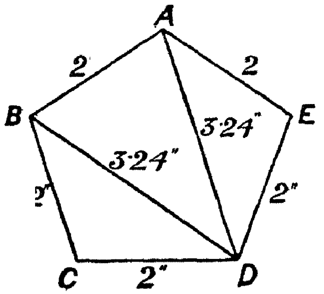


FIG. 68.

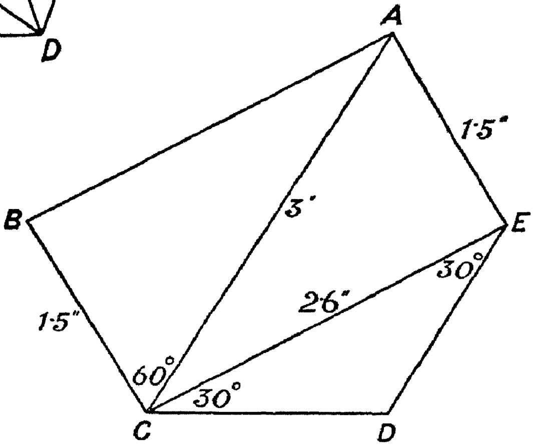


FIG. 69.

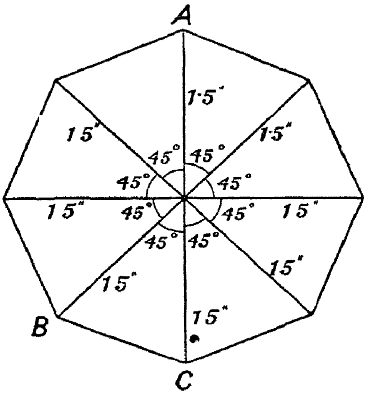


FIG. 70.

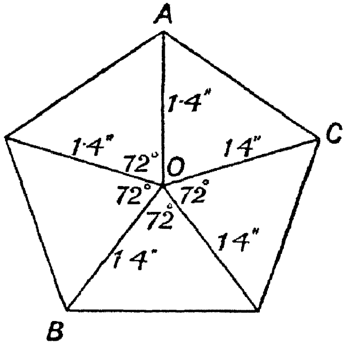


FIG. 71.

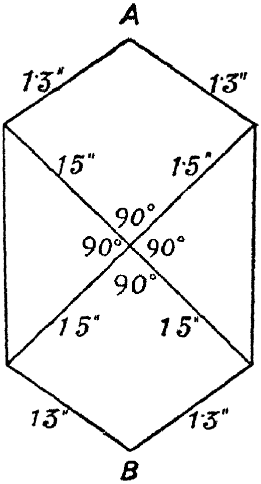


FIG. 72.

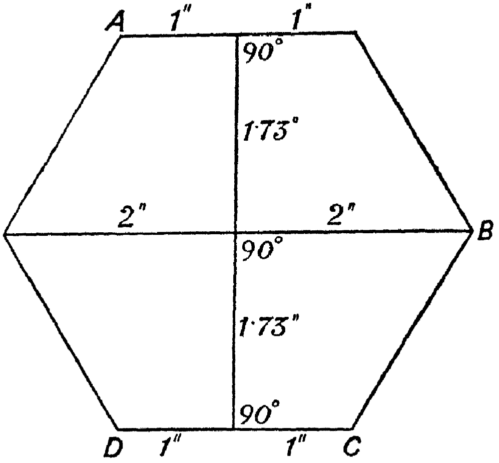


FIG. 73.

23. In your copy of Fig. 72 measure AB.

24. In your copy of Fig. 73 measure $\angle ABC$ and $\angle ADB$.

28. **On Symmetry.**—Take a sheet of paper and fold it in "two;" keeping it folded, cut it with the scissors into any shape you like, without cutting away much of the crease. If you now unfold the paper and spread it flat, you have a figure divided by the crease into two halves which exactly correspond to one another. If Fig. 74 represents the shape into which the paper is cut when folded, then Fig. 75 represents the figure produced when the paper is unfolded. Such a figure is called a **symmetrical figure**, and the line AB is called the **axis of symmetry**. One half of the figure is said to be symmetrical to the other half *with respect to* the axis.

If you draw a circle on paper, fold it so that the crease goes through the centre, and then cut the folded paper along the semicircle which shows on one side, you will find that your cut also passes along the other semicircle. Thus a circle is symmetrical with respect to any diameter as axis.

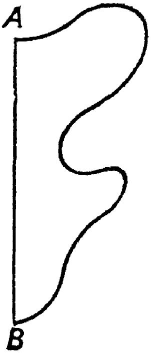


FIG. 74.

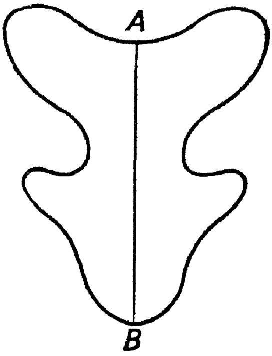


FIG. 75.

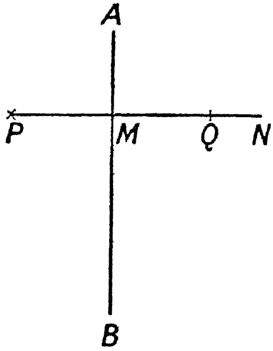


FIG. 76.

Problem XVI.—*To find a point symmetrical to a given point with respect to a given axis.*

Let us find a point symmetrical to the point P with respect to the axis AB (Fig. 76) :—

Draw $PM \perp$ to AB (produced if necessary) and meeting AB at M; produce PM to N, and from MN cut off MQ equal to PM. Then Q will be symmetrical to P with respect to AB; that is to say, if the figure is folded over so that AB is the crease, then the point P will fall on the point Q.

NOTE.—If P is on the axis AB it is then symmetrical to itself.

Problem XVII.—*To draw a line symmetrical to a given straight line with respect to a given axis.*

Let us draw a line symmetrical to PQ with respect to the axis AB (Fig. 77) :—

Using the preceding construction find the point R which is symmetrical to the point P with respect to the

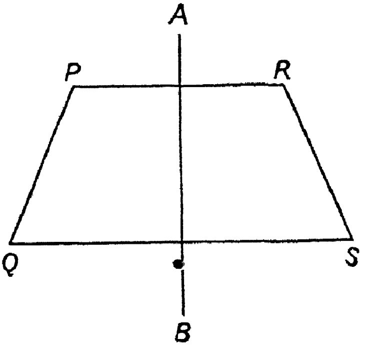


FIG. 77.

axis *AB* and the point *S* which is symmetrical to the point *Q* with respect to the axis *AB*. Join *RS*, which will be the required line; that is to say, if the figure is folded along the crease *AB*, the line *PQ* will coincide with the line *RS*.

NOTE 1.—If *P* lies on *AB* it is only necessary to find the point *S* which is symmetrical to *Q*; *PS* will then be symmetrical to *PQ*.

NOTE 2.—If *P* and *Q* lie on *opposites sides* of the axis *AB*, then the line *PQ* cuts *AB*; the method of finding the symmetrical line *RS* is the same as before. It will be found in this case that *PQ* and *RS* intersect at a point on *AB*.

NOTE 3.—In Fig. 77 it is worth noting that *PQ* and *RS* *produced* would intersect on *AB*.

NOTE 4.—In order to draw a figure symmetrical to a given figure, we must draw lines and points symmetrical to each of the lines and points which make up the given figure.

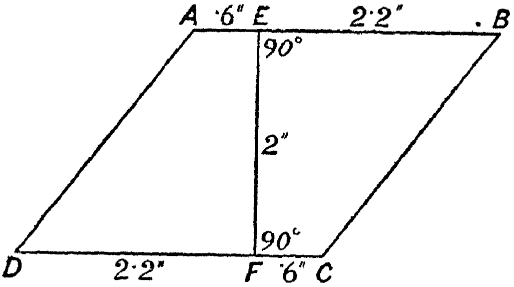


FIG. 78.

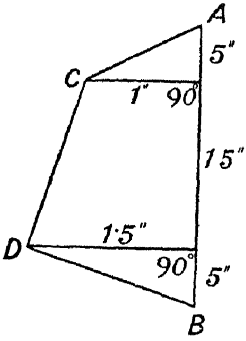


FIG. 79.

In working Exercises XIV. note particularly that when a figure is symmetrical the one half of the figure is similar to the image of the other half which would be formed by a mirror placed at the axis of symmetry, and held at right angles to the paper.

EXERCISES XIV.

1. Copy Fig. 79 to the dimensions given; complete the figure making it symmetrical to the axis *AB*. If *E* is the point symmetrical to *C*, and *F* the point symmetrical to *D*, measure *CF* and *DE*.

2. Copy Fig. 79 to the dimensions given; complete the figure, making it symmetrical to the axis *CD*. If *G* is symmetrical to *A*, and *H* to *B*, measure the distances *AH* and *BG*.

3. Copy Fig. 79 to the dimensions given; complete the figure making it symmetrical to the axis *BD*. If *P* is symmetrical to *A*, and *Q* to *C*, measure the distances *AQ* and *CP*.

4. Copy Fig. 78 to the dimensions given; complete the figure making it symmetrical to *EF*. If *P* is the point where *AD* intersects the line symmetrical to *BC*, and *Q* the point where *BC* intersects the line symmetrical to *AD*, measure *PQ*.

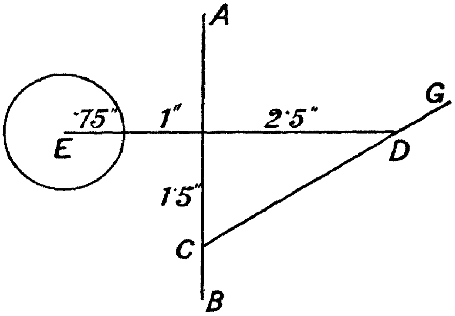


Fig. 80.

5. Copy Fig. 80 to the dimensions given; *E* is the centre of the circle. Complete the figure making it symmetrical to *AB*. Measure the distance between the two points in which the circle cuts the line symmetrical to *CG*; also the distance between the two points in which *CG* cuts the circle symmetrical to the given circle.

6. Copy Fig. 81 to the dimensions given; complete the figure making it symmetrical to the axis *AD*. In the result what point is symmetrical to *F* and what to *C*? What line is symmetrical to *FE* and what line to *AB*?

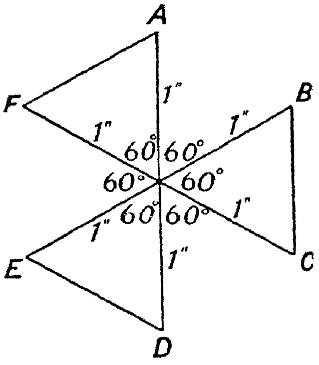


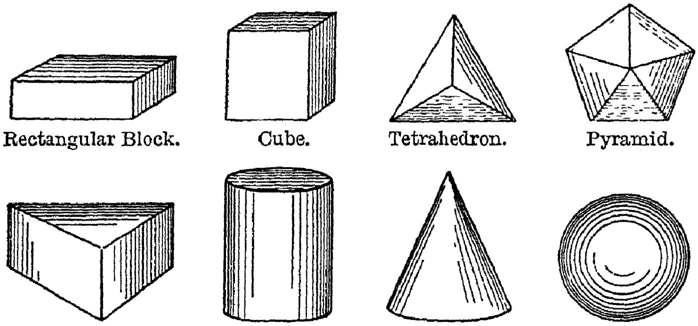
Fig. 81.

7. (i) What axes of symmetry has a rectangle? (ii) Has a parallelogram which is not a rectangle any axes of symmetry?

8. (i) What axes of symmetry has a square? (ii) Has an equilateral triangle any axes of symmetry?

29. **Geometrical Solids.**—All the geometrical figures which we have mentioned up to the present—circles, triangles, rectangles, etc.—have been flat or **plane figures.** Such figures have length and breadth, but *no thickness.*

We shall now learn the names and shapes of the simpler **solid figures,** *i.e.* figures which are not flat, but have length, breadth, *and thickness;* for instance, figures shaped like a brick, a ball, a round box, or a lead pencil.



Rectangular Block. Cube. Tetrahedron. Pyramid.

Wedge. Cylinder. Cone. Sphere.

FIG. 82.

These solid figures cannot be drawn on paper as the flat figures have been, and all that can be done is to draw a picture which will suggest their shape. Thus the picture of a round ball is merely a flat circle without thickness, and this picture is sometimes *shaded* to suggest the thickness. To represent the solids accurately we must make models of them, and we shall therefore show how to do this.†

The solids which we shall study are the *rectangular block, cube, tetrahedron, pyramid, wedge, cylinder, cone* and *sphere.* (See Fig. 82.) Note that the tetrahedron and pyramid are represented as seen from above.

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† Models of these solids may be made in plasticine or modelling clay, or cut out from paper or cardboard according to the instructions given. A set in wood may be obtained from Messrs. Clive & Co., 48 Southampton Row, W.C., but it is desirable for the teacher to have one larger set, and for the class to have occasional practice in drawing these from various points of view.

30. **The Rectangular Block.**—This is a solid figure in the shape of a brick, or an ordinary match-box. A picture of this figure is given in Fig. 83.

To construct a model of a figure of this kind draw Fig. 84 accurately to the given dimensions on *stiff* paper or cardboard. (Put the letters in the *same positions* as in Fig. 84.)

Now cut the figure out, cutting along the thick lines. Fold the figure to make creases at all the thin lines; if cardboard is used, first run the point of a penknife along the thin lines (taking care not to cut completely through the cardboard) so that the cardboard will fold accurately at these lines. The various parts of the figure can now be bent into position to make the model represented in Fig. 83, and fastened by means of gummed paper.†

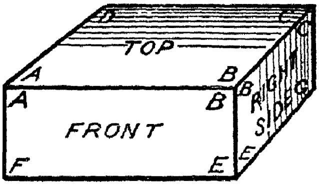

FIG. 83.

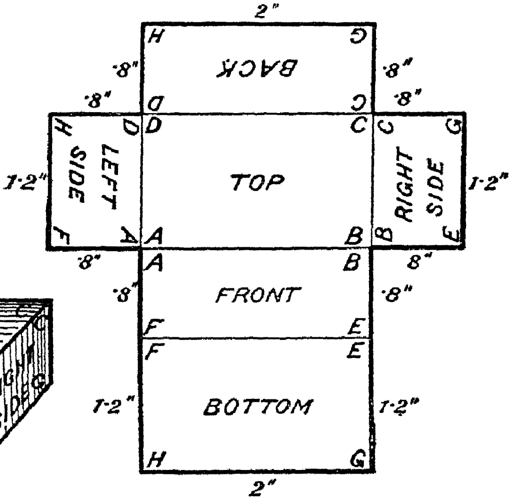

FIG. 84.

This model is not accurate in one respect, viz. it is hollow, whereas the rectangular block should not be hollow.

---

† All these models of solid figures should be kept for further use. See Chapter I.

Fig. 84 consists of six rectangles; these form the six **faces** of the block, viz. the top, bottom, front, back, right side and left side. In Fig. 83, three of these six faces are not shown, being "out of sight."

The sides of these rectangles are called **edges** of the block. Thus in Fig. 83 *AF, AB,* and *BC* are three of the edges of the block.

The points at the ends of these edges are called **corners** of the block. Thus, in Fig. 83, *A, B, C,* and *F* are four of the corners of the block.

These terms—*corners*, *edges*, and *faces*—apply to other geometrical solids. Remember that *corners are points, edges are lines*, and *faces are planes.*

The rectangular block may be made to any dimensions, but each face must be a rectangle, and opposite faces must be equal rectangles.

Note that in the rectangular block there are many pairs of parallel edges; thus *AF* and *BE* are parallel, and *AF and CG are parallel.*

Again in the rectangular block opposite faces are *parallel planes.* Thus the top and bottom are opposite faces; and if these two faces are produced *as planes* in all directions they will never meet.

If the model is correctly constructed from Fig. 84 the three letters at any one corner should be the same, *e. g.* at one corner there should be three *B*'s, at another three *F*'s, and so on.

31. **The Cube.**—The cube is a rectangular block in which all the edges are equal, and hence all the faces are squares.

To make a model of a cube, draw a figure similar to Fig. 84, altering each of the given lengths to 1·5″. Build up the model from this figure as before.

EXERCISES XV.

Using the model of the rectangular block made from Fig. 84:—

1. Count the number of (i) faces, (ii) edges, (iii) corners.

2. Name the edges which meet at the corner (i) *A,* (ii) *G,* (iii) *E.*

3. Write down all the edges which are equal to the edge (i) *AB,* (ii) *DH,* (iii) *EG.*

4. Write down all the edges which are parallel to the edge (i) *AD,* (ii) *CG,* (iii) *EF.*

5. If the block is standing on a table, how many edges of the block lie along the table, and how many corners touch the table?

6. Which of the other faces is parallel to the left side, and which to the back?

7. The figure *DHEB* is a rectangle which passes through the middle of the block. Measure *DB* and *BE* and hence draw on a piece of paper a rectangle exactly equal to the rectangle *DHEB*. Use this rectangle to measure *DE* and *BH* (which are two "diagonals" of the block) and the parts into which they divide one another.[†]

8. Draw on paper a rectangle which is exactly equal to the rectangle *DCEF*. Use this rectangle to measure the "diagonals" *DE* and *CF*, and the parts into which they divide one another.

9. Draw on paper a rectangle which is exactly equal to the rectangle *DAEG*. Use this rectangle to measure the "diagonals" *DE* and *AG*, and the parts into which they divide one another.

10. What do you infer (from Questions 7, 8, and 9) with regard to the four diagonals of a rectangular block? Note that the diagonal *DE* occurs in each question.

11. Working as in Question 7, find the length of a diagonal of the cube constructed from the directions given in § 31.

12. Draw a sketch of a brick standing on one end.

32. **The Tetrahedron.**[‡] — This is a figure bounded by four faces, each of which is a triangle.

A model of a tetrahedron can be constructed from Fig. 85.[§] The middle triangle forms the bottom or **base** of the tetrahedron, and the three outer triangles must be bent over till the three points marked *A* come together. Fasten the model together with gummed paper.

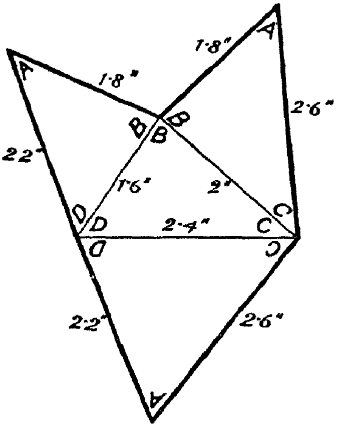

FIG. 85.

[†] The diagonals of the *faces*, such as *AC* and *BF*, are *not* diagonals of the *block*. The former lie entirely on the surface of the block; the latter *pass through* the block.

[‡] From Greek words meaning *four faces*.

[§] In this and in all similar cases copy all the letters on the figure, but none of the numbers.

Fig. 86 is a picture of this tetrahedron held in such a position that the three upper faces are visible at once.

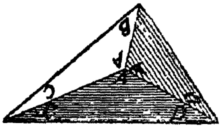

FIG. 86.

If all the faces are equilateral triangles the figure is called a **regular tetrahedron.**

33. The Pyramid.—

DEFINITION.—**A figure bounded by more than four straight lines is called a polygon.**

Thus Figs. 68, 69, 71 show polygons with five sides; Fig. 70 shows a polygon with eight sides.

Imagine any polygon *ABCDEF* drawn on a table (Fig. 87) and imagine some point *V* (in the air) above this polygon; this polygon we will call the base, and this point *V* we will call the vertex. Now suppose this vertex *V* is joined by straight lines (the dotted lines in Fig. 87) to each of the angular points of the base. Then the sides of the base and these lines drawn from the vertex form the outline of a geometrical solid which is called a pyramid.

Fig. 88 represents a pyramid viewed from a point above the vertex. The base of a pyramid may be any polygon, or a quadrilateral, or a triangle, but the other faces of the pyramid (called the sides) are all triangles.

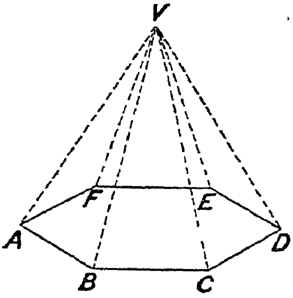

FIG. 87.

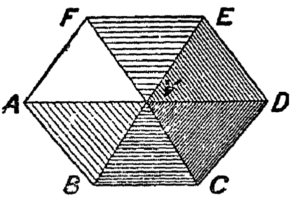

FIG. 88.

If the base is a triangle the pyramid is a tetrahedron. Hence a tetrahedron may be called a *triangular pyramid.*

In each of the pyramids of Egypt the base is a square, and the vertex is directly above the centre of the base. These pyramids are huge tombs built (in stone or brick) by various kings of Egypt thousands of years ago. Some of them are built with very great mathematical accuracy and architectural skill.

An accurate model of the Great Pyramid[†] can be constructed from Fig. 89. The triangles must be bent over till all the points marked *V* come together. *ABCD* should be a square of side 2·2″, and the remaining sides of the triangles should measure 2·1″ each.[‡]

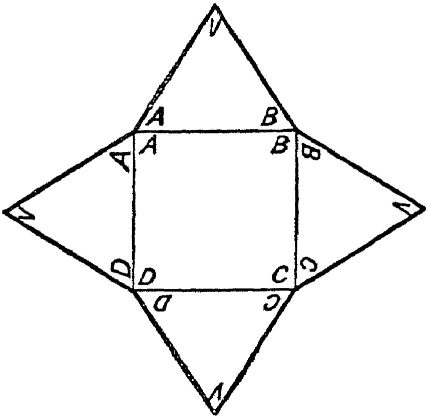

FIG. 89.

34. **The Wedge.**—Suppose a triangle *ABC* drawn on the table (Fig. 90); suppose three pins of equal lengths (*DA, EB, FC*) fixed into the table in an upright position at the angular points of the triangle. Now imagine the heads of these pins joined by straight lines (dotted in Fig. 90); these dotted lines form a triangle *DEF* equal to the triangle *ABC.*

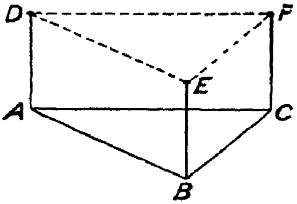

FIG. 90.

The lines of the whole figure form the outline of a geometrical solid called the **wedge** (or *triangular*

[†] Built by Khufu (Cheops) probably about 3900 B.C. The length of each side of the base is 250 yards.

[‡] 2·09″ would be more accurate.

prism†). Two faces are equal triangles of any shape; the remaining three faces are rectangles.

A model of a wedge can be constructed from Fig. 91.

A piece of cheese is generally cut (roughly) in the shape of a wedge, though the surface formed by the rind curves a little.

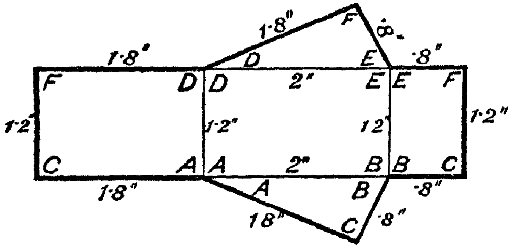

Fig. 91.

EXERCISES XVI.

1. How many edges and how many corners have the following solids? (i) Tetrahedron, (ii) Square pyramid, (iii) Pyramid having a five-sided polygon as base, (iv) Pyramid having an eight-sided polygon as base, (v) Wedge.

2. If a tetrahedron is standing on a table how many corners touch the table?

3. If a square pyramid is standing on a table (on its base) how many corners touch the table, and how many edges lie along it?

4. What is the answer to Question 3 when the pyramid is lying with one of its sides in contact with the table?

5. In the wedge constructed from Fig. 91 :—
   (*a*) Which two faces are parallel?
   (*b*) Mention a set of three parallel edges.
   (*c*) Mention three other sets of two parallel edges.

6. Construct the outlines of a tetrahedron by means of lines drawn on a sheet of paper and three pins.

7. Construct the outline of a square pyramid by means of lines drawn on a sheet of paper, and four pins.

8. Measure the height of your model of the Great Pyramid constructed from Fig. 89. [Measure AC; then draw △ ACV on paper. Draw $VO \perp$ to AC; VO will be the height of the pyramid.]

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† If a polygon is used instead of the triangle ABC, with equal pins at each angular point, and so on, the resulting figure is the outline of a polygonal prism.

35. **The Cylinder.**—Fig. 92 is a picture of two cylinders. A drum, a round box, an unsharpened lead-pencil, a roll of carpet, a penny, and a pile of pennies are all roughly in the shape of cylinders; the lead-pencil is a very tall cylinder, and the penny a very short cylinder.

The easiest way to construct a model of a cylinder is to take a long strip of paper (of uniform width) and roll it up into the shape of a roll of carpet.

The base of a cylinder is a circle, and the top is an equal circle. The remainder of the surface of a cylinder is usually called the **curved surface**; this curved surface stretches from the one circle to the other, and is bounded by the circumferences of these two circles.

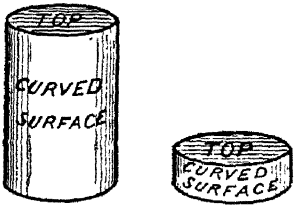

Fig. 92.

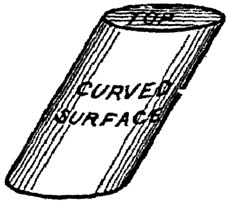

Fig. 93.

In the roll-of-paper model of the cylinder this curved surface is represented by the outside layer of paper, that is by the portion of paper which was last wrapped round the roll.

The line which joins the centre of the top of a cylinder to the centre of the base is called the **axis** of the cylinder.

The kind of cylinder shown in Fig. 92 is the simplest type of cylinder, and should really be called the *right circular cylinder*. Fig. 93 shows *a skew circular cylinder;* in this case the axis of the cylinder is on the slant instead of being upright as in Fig. 92; but the top and base are still equal circles. In this case the curved surface slants in the same direction as the axis. If the top and base are not circles but other curves then the figure is still called a cylinder, but is not a *circular* cylinder.

36. **The Cone.**—Suppose a circle drawn on the table, with centre C (Fig. 94); this circle is to be called the **base**. Imagine a point V in the air exactly above the centre C; this point V is to be called the **vertex.** Now imagine that V is joined by straight lines to every point on the

circumference of the circle; these straight lines will form a **curved surface**, and the solid figure contained between this curved surface and the base is called a **cone.** *VC* is the **axis** of the cone.

Thus the cone, like the cylinder, stands on a circular base; but the cylinder is the same thickness from base to top, whereas the cone tapers away to a point at the vertex.

The lower end of a cricket-stump and the sharpened end of a pencil are good examples of cones.

To construct a model of the cone cut out a piece of paper in the shape of Fig. 95, which is a portion of a circle whose centre is *V*. Bend the paper round, fastening the edge *VA* to the edge *VB* with gummed paper; this will make the curved surface of the cone, *V* being the vertex. A circle of radius one inch will make the base, and can be fastened to the curved surface by gummed paper.

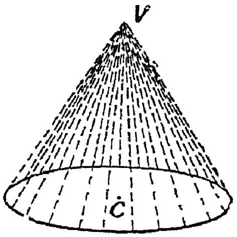

Fig. 94.

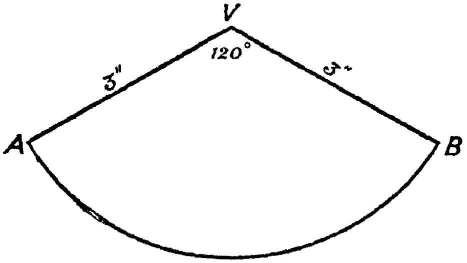

Fig. 95.

Fig. 94 represents a *right circular cone.* If the axis *VC* were not upright it would be a *skew circular cone,* or *oblique cone.*

37. **The Sphere.**—A geometrical solid in the shape of a perfectly round ball is called a **sphere.**

The sphere is bounded by one curved surface. All points on this surface are at equal distances from a certain point within the sphere called its **centre.**

Any line joining the centre of a sphere to a point on its surface is called a **radius.** Obviously all radii of a sphere are equal.

Any line passing through the centre of a sphere and terminated at both extremities by the surface of the sphere is called a **diameter.** Obviously any diameter is twice the length of a radius.

A good model of a sphere can be easily constructed by rolling a lump of plasticine or modelling clay between the hands until it is perfectly round. For most purposes a round glass marble of large size will serve.

EXERCISES XVII.

[In these exercises the words *cylinder* and *cone* denote the simple forms of these figures, viz. the right circular cylinder and cone.]

1. Suppose that the rectangle *ABCD* (Fig. 96) can swing round the side *BC* like a door round its hinge-line, and imagine the rectangle to make one complete revolution.

(*a*) What plane figure is traced out by the side *AB ?*
(*b*) What plane figure is traced out by the side *CD ?*
(*c*) What solid figure is traced out by the whole rectangle*?*
(*d*) What figure is traced out by the side *AD ?*

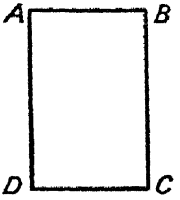

FIG. 96.

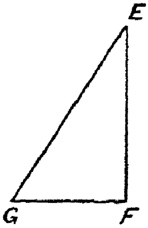

FIG. 97.

2. Suppose that the right-angled triangle *EFG* (Fig. 97) can swing round the side *EF* like a door round its hinge-line, and imagine the triangle to make one complete revolution.

(*a*) What plane figure is traced out by the line *FG* ?
(*b*) What solid figure is traced out by the triangle *EFG* ?
(*c*) What figure is traced out by the side *EG* ?

3. What solid figure is traced out by a semicircle rotating about the diameter which forms its base ?

4. A cylinder of wood is cut through so that the cut surface is a plane parallel to the base of the cylinder. What would you expect the shape of the cut surface to be ?

5. A cone is cut through so that the cut surface is parallel to the base of the cone. What would you expect the shape of the cut surface to be ?

6. A sphere is cut through so that the cut surface is a plane. What would you expect the shape of the cut surface to be (*a*) if the cut passes through the centre of the sphere, (*b*) if the cut does not pass through the centre ? [Imagine a perfectly round turnip sliced through by a knife.]

CHAPTER I†.

THE SCIENCE OF GEOMETRY.

38. In the preceding portion of this book we have been concerned only with accurate and intelligent drawing, that is with *Practical Geometry.* We have learnt several easy geometrical constructions—such as how to draw a line perpendicular to a given line; also we have noticed several interesting properties of geometrical figures—for example, that the three angles of any triangle are together equal to two right angles, that the diagonals of a parallelogram bisect each other, etc.; but we have not tried to discover the reasons for these properties or constructions.

In the remainder of this book we shall learn many more constructions and many more properties of geometrical figures; but we shall in all cases *learn the reasons* for these constructions and properties. We shall find that if a few obvious facts are assumed (*i. e.* taken for granted), it is possible to prove all the known properties of geometrical figures by a long series of careful arguments.

This logical discussion of geometrical properties forms the *Science of Geometry.*

39. **Theorems and Problems.**—A **theorem** is the statement of some geometrical truth.

Thus the statement, "The three angles of any triangle are together equal to two right angles" is a theorem.

A **problem** is the statement of some required geometrical construction, such as:—

"To draw a line parallel to a given line through a given point."

The logical discussion of Geometry consists chiefly of a large number of separate sections called **Propositions.**

The object of each proposition is either (i) to prove the truth of a theorem, or (ii) to state the construction required for a problem, and to prove that this construction is correct.

A **corollary** to a theorem is a second theorem whose truth is easily deduced from the first.

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† See the note on this chapter in the Preface, p. x.

40. **Fundamental Concepts.**—All accurate knowledge is based upon certain *fundamental concepts*, that is to say upon certain ideas which cannot be explained by reference to any more simple ideas.

If a man is asked what he means by a house, he will describe it as made up of walls, floors, ceilings, windows, etc. Thus the idea of a house *is not* a fundamental concept, as it can be explained by reference to other more simple ideas.

If however he is asked what he means by redness, he can only say that he knows a red object when he sees it, but that he cannot explain his idea of redness by reference to any more simple ideas. Thus the idea of redness *is* a fundamental concept.

41. **Space.**—The Science of Geometry is based upon a set of fundamental concepts which may be described as *ideas of the existence of, and the nature of, space.* These ideas include such concepts as *position*, *solidity*, *distance*, *direction*, and many others which are more or less closely connected with these.

It must however be admitted that it would be extremely difficult to draw up an accurate and complete list † of those ideas which should be classed as the fundamental concepts of Geometry. We have a large number of elementary ideas on space which are closely related to one another, and many of which can be explained more or less satisfactorily by means of the others, but it is scarcely possible to select from these a group of concepts which can be regarded as essentially simpler than the rest, and by means of which the rest can be adequately explained. Fortunately the question is not of much practical importance. It is important, however, that the beginner should realize that Geometry, like any other branch of accurate knowledge, is *based upon a certain group of ideas which cannot be explained by reference to any simpler ideas*, and a knowledge of which must therefore be taken for granted.†

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† So far as the authors of this book can discover Professor Bain is the only logician who has attempted to deal with this question. In *Logic, Induction* (p. 206) he gives the following list of fundamental concepts, with which most mathematicians would probably disagree, both as to what it includes and what it omits :—

Point, *line* or *length*, *straight* (with *bent*), *angle*, *surface*, and *solid*. He adds that the notion of *straight* is necessarily implied in that of *length* as "it is only the straight line that is synonymous with length," and that the ideas of *direction* and *parallelism* are implied in that of *angle*. The definitions ordinarily given of *line* and *plane* he regards as "convenient testing peculiarities."

G. T. P.

42. **Definitions.**—By means of the fundamental concepts we can explain the meaning of all other geometrical terms, such as point, line, triangle, etc.

These terms must be explained as accurately as possible, so that there is no fear of mistaking their meaning. For if we are not quite sure of the meaning of the terms we are using, our arguments will obviously be worthless.

The explanation of the exact meaning of any term is called a ***definition.***

Definitions must be *complete* (*i. e.* they must contain enough) and must not be *redundant* (*i. e.* they must not contain too much). For example :—

The definition "A triangle is a plane figure contained by three lines" is incomplete ; for it omits to state that the lines must be straight lines.

"A triangle is a plane figure with three angles, contained by three straight lines" is a redundant definition ; for it is unnecessary to mention the three angles, since all plane figures contained by three straight lines have three angles.

"A triangle is a plane figure with three angles contained by three lines" is both incomplete and redundant.

---

† Note that the young child does not first learn the fundamental concepts, and afterwards derive the other concepts from the fundamentals; the process is quite different At first he receives certain vague impressions through seeing, handling, and moving objects, and through moving himself. After a while, from these impressions he obtains fairly accurate ideas of rest, motion, shape, etc. ; and from these again, he at last obtains with more or less accuracy the ordinary ideas of space.

The definitions of the various terms will be given from time to time as need arises. In all cases they should be learnt by heart. The two most elementary definitions are those of a point and a line.†

DEFINITION.—**A** *point* **has position, but has no parts and no magnitude.**

DEFINITION.—**A** *line* **is the path traced out by a moving point; it has length, but no breadth and no thickness.**

These definitions have already been given (see §§ 1, 2). In the Introductory Course we were considering only points and lines lying on a flat surface, but these definitions apply equally well to points and lines in space.

43. **The Straight Line.**—A straight line can be defined in many different ways, and it does not matter very much which definition is adopted. No one ever learnt the meaning of the word "straight" from any definition.

From the logical point of view the best definition will be that which is most useful for geometrical argument. Now in Theoretical Geometry the most important property of straight lines is the fact that any two straight lines can be fitted together in such a way that *no space is left between them*; or to express the same idea in different language, that any straight line can be made to "lie along" any other straight line. This property suggests the following

DEFINITION.—*Straight lines* **are lines such that if ANY two points of one are made to coincide with two points of another there is never any space included between the two lines.**‡

---

† Even if we assume with Bain that *point* and *line* are fundamental concepts, it will still be necessary to define them. Our definitions in this case must be regarded as statements whose object is to transform the popular concepts of point and line into the more definite concepts required in geometrical science.

‡ Euclid (the famous Greek mathematician who taught at Alexandria in Egypt about 300 B.C.) did not use this property as a definition, but embodied it in the *axiom* (see § 49) "Two straight lines cannot enclose a space." If any other definition of a straight line is adopted this axiom of Euclid's must be retained.

Note that this definition assumes the concept of *rigidity*; that is to say, it assumes that lines can be moved about in space without change of size or shape.

Illustration.—Fig. 98 shows two straight lines. In Fig. 99 the points A and B of the first straight line are made to coincide with the points C and D of the second, and there is *no space* between the lines.

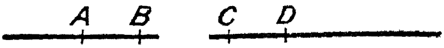

Fig. 98.

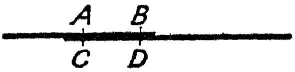

Fig. 99.

Fig. 100 shows two curved lines. In Fig. 101 the points A and B of the first curved line are made to coincide with the points C and D of the second, and it happens that again there is no space between the lines. In order to show that these lines do not satisfy the definition given above, reverse one of them, as in Fig. 102; the points A and B again coincide with the points C and D, but there is now a space between them. The last figure shows the importance of the word "never" in the definition.

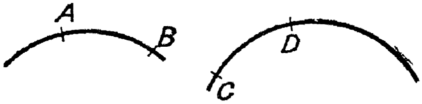

Fig. 100.

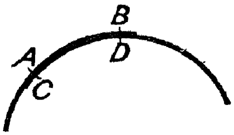

Fig. 101.

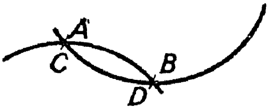

Fig. 102.

The lines should of course have no thickness. In these figures, in order to show the two different lines clearly they have been drawn very thick; but it should be noticed that in Figs. 99 and 101 if the lines were perfectly thin they would *coincide* along a certain portion of their length.

DEFINITION.—**Lines which are not straight, or made up of straight lines, are called *curved lines* or curves.**

44. Alternative Definitions of the Straight Line.—The following alternative definitions of a straight line are worthy of notice.†

1. A *straight line* is the shortest distance between two points. (Archimedes' definition.)

2. A *straight line* is a line which lies evenly between its extreme points. (Euclid's definition.)

3 A straight line is a line such that all its parts point in the same direction.

4. A *straight line* is such that if its two extremities are fixed and the line is turned about these two extremities as pivots, no point on the line will change its position.

EXERCISES XVIII (*Oral*).

1. What is a *Geometrical Theorem?*

2. State any Geometrical Theorems which you know.

3. What is a *Geometrical Problem?*

4. Mention any Geometrical Problems for which you have learned the constructions.

5. What is a *Fundamental Concept?*

6. Which of the following are Fundamental Concepts—Parallelogram, space, quadrilateral, position, triangle, circle, distance?

7. What is a *Definition?*

8. Define a *Point*, and a *Line.*

9. Define a *Straight Line.*

10. Which of the following definitions are redundant and which incomplete?—

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† The first definition is based on the principle of the stretched-thread test for straightness given in the Introductory Course (§ 2). It is obvious that the stretched thread gives the *line of shortest distance* between its extremities. This is an elementary and important property of the straight line, but it is not of such fundamental importance in Theoretical Geometry as the "coincidence" property.

The second definition either has no very accurate meaning, or else means that if a straight line is viewed from one end it appears to be a point (compare the second test for straightness given in the Introductory Course). This ultimately reduces to the statement that a straight line is the path of a ray of light—which is not necessarily true, and in any case is not suitable for a definition.

The third definition may appear satisfactory, if we regard direction as a fundamental concept; but the concept of direction is itself vague and incomplete without the concept of straightness.

The fourth definition is based on the principle of the third test for straightness given in the Introductory Course (§ 2).

(*a*) A circle is a plane figure bounded by one curved line.

(*b*) A quadrilateral is a plane figure with four angles bounded by four straight lines.

(*c*) A diameter of a circle is a straight line passing through the centre terminated at both ends by the circumference and equal in length to two radii.

(*d*) A rhombus is a plane figure bounded by four equal lines.

11. Show that each of the following definitions is incorrect :—

(*a*) A point is a very small particle.
(*b*) A line is the path traced out by a moving particle.
(*c*) A point is that which has position.
(*d*) A line is a row of points.
(*e*) A line is that which has length.
(*f*) A curved line is a line which is not straight.

45. **Surface.**—The *surface* of a solid means the boundary which *separates the solid from the space outside it.*

Fig. 103 represents a brick (or a rectangular block).

The surface of the brick consists of six parts, which may be described as the top, the bottom, the front, the back, the right side, and the left side. In Fig. 103 the bottom, the back, and the left side are out of sight. These six parts of the surface are sometimes called the six *faces* of the brick ; and it may be noted that each face is a rectangle † (see § 26). At present, however, the important point is that these six faces form the boundary which separates the material of the brick from the air outside it.

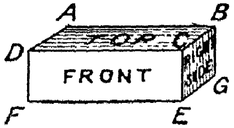

FIG. 103.

The six faces of a brick are all flat ; but the surface of a round box (or cylinder ; see § 35) consists of two flat faces, the top and the bottom, and one curved portion (see Fig. 104). The surface of a ball (or sphere ; see § 37) is entirely curved.

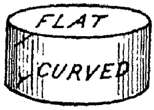

FIG. 104.

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† If we measure the angles *ABC, GBC* in Fig. 103 we shall find that they are not right angles. Thus it is quite obvious that the figures *ABCD, BGEC in this diagram* are *not* rectangles ; they merely *represent* rectangles as seen "in perspective." We cannot use the picture of a solid body to measure the angles and lines of the solid body itself.

The surface of a solid has no thickness. Thus if a point in the solid is only one-thousandth of an inch from the space outside the solid, we should say that it is not *on* the surface, but one-thousandth of an inch *inside* it.

A surface need not completely enclose a space. Thus we may conceive of a surface dividing a room into two parts, like an exceedingly thin wall; note too that such a surface could be of various shapes, flat or otherwise.

DEFINITION.—**A surface † is the boundary between two regions of space; it has length and breadth, but no thickness.**

The boundary of a surface is always a line, or a series of lines. Thus the front of the brick in Fig. 103 is a rectangle, and this rectangle is bounded by the four lines *DF, FE, EC, CD.* The boundary of the top of a round box is the circumference of a circle (see § 35).

The surface of a ball is not divided into different parts, but any portion of it would be regarded as a surface in the geometrical sense of the word, and any portion would be bounded by a line or a series of lines.

The extremities (or boundaries) of a line are points. Thus the "edge" *DC* of the brick is a line, and the extremities of this line are the points *D, C,* which are "corners" of the brick.

Geometry may be defined as the Science which studies the relations and properties of points, lines, surfaces, and solids.

46. **Dimensions.**—A *line* is said to have *one dimension,* viz. length. A *surface* is said to have *two dimensions,* viz. length and breadth. A *solid* is said to have *three dimensions,* viz. length, breadth, and thickness.

A moving point traces a line.
A moving line will in general trace a surface.
A moving surface will in general trace a solid.

47. **The Plane.**—It is *not* possible to draw a straight line on the surface of a ball. The straight line which joins any two points on the surface of a ball necessarily passes inside the surface.

---

† Another name for surface is *superficies.*

It *is* possible to draw a straight line on the curved surface of a round box (or cylinder). If the box is standing upright in the usual manner and if two points are taken on the curved surface such that one is directly below the other, the straight line joining these points will lie entirely on the surface (see Fig. 104). With this exception the straight line joining any two points on the curved surface of the round box will pass inside it.

If however we take *any* two points on the top surface of the round box, the straight line joining these two points lies entirely on the surface. Such a surface we call *flat*, or, in geometrical language, *plane*.

DEFINITION.—**A plane surface (or plane) is a surface such that the straight line which joins any two points upon it lies entirely on the surface.**

To test whether any surface is an exact plane, see whether the straight edge will fit accurately to the surface at every part. Note that this test is a direct application of the definition.

48. **Intersections.**—Two lines when they cross one another do so at *points*. These points are called the **crosses** or **intersections** of the lines.

If both lines are straight they can only meet in *one* point. If either line (or both) is not straight they may intersect in more than one point. (See § 4).

If two straight lines do not meet, the point at which they would meet if *produced* (i. e. lengthened) may still be called the *cross* of the lines.

The straight line joining two points A and B is called the "**join** of A and B."

The intersections of a line and a surface are *points*.

A piece of paper pierced by a thin wire gives a good illustration of a surface (viz. the paper) intersected by a line (viz. the wire). The holes which the wire makes in the paper represent the points of intersection.

If the line is straight and the surface is a plane they can intersect in one point only. Otherwise they may intersect in more than one point.

If the wire or the paper may be bent, it is possible to pierce the paper several times with the one wire.

The intersections of two surfaces are *lines*.

A strip of paper pushed through one or more slits in another piece of paper gives a good illustration of two intersecting surfaces. The slits represent the lines of intersection.

If both surfaces are plane they can intersect only in *one straight line*.

One flat piece of thin metal cannot be pushed through a slit in another flat piece of thin metal (without bending either piece of metal) unless the slit is straight. Also the first piece of metal cannot be pushed through more than one slit in the second piece at the same time without bending it.

In Fig. 103 the top and front are two planes whose intersection is the edge *DC;* the front and right side are two planes whose intersection is the edge *CE*.

49. Axioms.—

DEFINITION.—**An** *axiom* **is a true statement whose truth cannot be proved by reference to any simpler statement.**

The two simplest axioms † in Geometry are :—

(1) *Things equal to the same thing are equal to one another.*

(2) *If equals be added to equals the wholes are equal.*

Note the following illustration of Axioms (1) and (2).

(1) A, B, C are three boys. If A is the same height as C, and B is the same height as C, then the first axiom states that A is the same height as B.

(2) If, in a year's time, it is found that these boys have grown exactly one inch each, the second axiom states that they are still the same height.

(3) A third axiom is required in the theory of parallels. We shall adopt Playfair's Axiom† (see § 64).

From the first two axioms other statements can be deduced which are usually given as independent axioms. Such are—

(4) *If equals be taken from equals the remainders are equal.*

---

† Mathematicians differ in regard to the fundamental axioms just as the differ in regard to the fundamental concepts (§ 40). The whole question is fully discussed in Russell's *Foundations of Geometry*.

(5) *If equals be added to unequals the wholes are unequal.*

(6) *If equals be taken from unequals the remainders are unequal.*

(7) *Things which are doubles of the same thing or of equal things are equal.*

(8) *Things which are halves of the same thing or of equal things are equal.*

One other statement is frequently included in the list of axioms :—

(9) *The same two points cannot be joined by two different straight lines;* i.e. *two straight lines cannot enclose a space.*

This last Axiom is directly derivable from the Definition of a straight line (given in § 43). If Definition 1 (of § 44) be adopted it means that a string tightly stretched between two points can follow one track only, presuming that there is no obstacle between the points.

We shall use these Axioms in the course of this book (without attempting to prove them).

In addition to these, there are many *Axiomatic Theorems* † which will be employed. A theorem, or statement, is described as Axiomatic when its truth is so obvious that it need not be proved.‡

Of these theorems the most important is :—

(10) *That any figure* § *in a plane may be moved from one part of the plane to another, or turned over and replaced in the plane without any change of shape.*

This axiomatic theorem *assumes* the concept of rigidity (see p. 68, lines 1–3), and states *two* fundamental properties of the plane, which are not stated in but are deducible from the definition of a plane. The proof by which these properties are deduced is, however, best deferred until a later stage, when the properties of the plane as a whole are studied.

The following are other instances of the Axiomatic Theorems which will be used in the logical course :—

---

† Properly Axioms 4–8 are Axiomatic Theorems.

‡ The exact logical proof of such is often far from easy.

§ *E.g.* a triangle or quadrilateral.

(11) If $A > B$ and $B > C$, then $A > C$.†
(12) If $A > B$, $B > C$, and $C > D$, then $A > D$.
(13) If $A > B$, $B = C$, and $C > D$, then $A > D$.

These theorems can all be proved, but proof is not necessary.

A re-statement, in other words, of a definition or of the idea involved in a fundamental concept must not be confused with an axiom. Thus the statement *Things which coincide or exactly fill the same boundary are equal* is not an axiom but merely a very incomplete definition of equality. So also *The whole is greater than its part* merely restates the fundamental ideas underlying the words *whole* and *part*.

Do not confuse Axioms with Fundamental Concepts. Axioms are *statements*; Concepts are *ideas* (such as position, length, distance, etc.).

We use Axioms in arguing about Concepts; the axioms are used as steps in the argument. The axioms are the tools, the concepts the material worked upon.

EXERCISES XIX.

Questions 1–14 *should be worked orally with the class.*

In Questions 1–10 use the models of geometrical solids constructed when working through §§ 30–36.‡

1. Which of the following solids are entirely bounded by plane surfaces?—Rectangular block, cube, tetrahedron, pyramid, wedge, cylinder, cone, sphere.

2. Which of these solids possess curved surfaces on which straight lines can be drawn?

3. In the rectangular block name the lines which are the intersections of the following pairs of plane surfaces:—(*a*) Top and left side; (*b*) right side and front; (*c*) left side and back.

4. In the square pyramid name the lines which are the intersections of the following pairs of plane surfaces:—(*a*) VAB, VBC; (*b*) $ABCD$, VAD.

5. In the rectangular block name all the edges and all the faces which meet at the corner F.

6. In the tetrahedron how many edges and how many faces meet at any one corner?

7. In the square pyramid how many edges intersect at the vertex, and how many faces intersect at the vertex?

8. In the wedge name all the edges which intersect at the corner E; also the faces which intersect at the corner E; also the faces whose intersection is the edge AD.

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† The sign > denotes *greater than* or *is greater than;* the sign < denotes *less than* or *is less than.*

‡ Students who have not constructed these models will have to omit questions 3, 4, 5, 8.

9. In the cylinder what is the shape of the intersection of the top face with the curved surface?

10. In the cone what is the shape of the intersection of the base with the curved surface?

11. The surface of still water is a plane. Given a basin of water and a thin sheet of metal, show how to illustrate:—

(i) That the intersection of two planes is always a straight line.

(ii) That the intersection of a curved surface and a plane may or may not be a straight line.

12. If a brick is partly immersed in still water, what will be the shape of the "water-line" on each face?

13. Why are the following definitions unsatisfactory?

(i) The surface of a solid is the outermost layer of the solid.

(ii) A plane surface is one on which straight lines can be drawn.

(iii) A straight line is a boundary of a plane surface.

(iv) A straight line is a line such that any portion of it can be made to coincide exactly with any other portion of equal length.

14. What relation between A and E can be derived from each of the following groups of statements? †

(i) $A=C,\ C=\frac{1}{2}E.$

(ii) $A=2B,\ B=\frac{1}{2}E.$

(iii) $A=2B,\ B=2C,\ C=E.$

(iv) $A>B,\ B=C,\ C=E.$

(v) $A=B,\ B<C,\ C=D,\ D<E.$

(vi) $A>B,\ B=C,\ C<E.$

15. Draw (from a model) a sketch of a straight piece of thin wire passing through a flat piece of thin cardboard. Indicate that portion of the wire which is hidden by the cardboard by means of a dotted line, and mark with a cross the point where the wire pierces the cardboard.

16. Draw (from a model) a sketch of a ring of thin wire passing twice through a flat piece of thin cardboard. Indicate the hidden portion of the wire by a dotted line and mark with crosses the points where the wire pierces the cardboard.

17. Draw (from a model) a sketch of a thin flat strip of metal passing through a thin flat sheet of cardboard, marking the straight line where the metal passes through the cardboard.

18. Draw (from a model) a sketch of a thin flat strip of metal passing twice through the curved surface of a cylinder and meeting the surface in a straight line at each intersection.

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† Illustrate each set of statements by straight lines representing A, B, C, etc. The lines for each set should be drawn by a pupil.

CHAPTER II.

ANGLES AND PARALLELS.

50. Angles, Straight Angles, and Right Angles.—

DEFINITION.—**A straight line turning in a plane about one extremity is said to describe an** ***angle.***

If a straight line turn from the position OA (Fig. 105) into the position OB it describes the angle AOB.

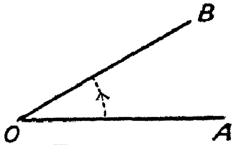

FIG. 105.

The two lines which indicate the initial and final position of the revolving line are called the **arms** of the angle, and the point about which the revolving line turns is called the **angular point.**

DEFINITION.—**If the two arms of an angle lie in a straight line and are on opposite sides of the angular point, then the angle is called a** ***straight angle*** **or** ***flat angle.*** .

Thus in Fig. 106 $\angle AOB$ and $\angle CQD$ are straight angles.

All straight angles are equal.

For if AOB and CQD are two straight angles, it is possible to apply the plane of the angle AOB to the plane of the angle CQD in such a way that the point O falls on the point Q and the line AB lies along the line CD. The two straight angles now coincide and are therefore equal.

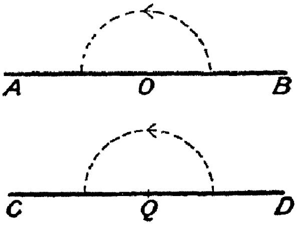

FIG. 106.

DEF.—**The half of a straight angle is called a** ***right angle.***

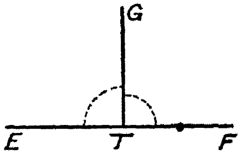

FIG. 107.

If one straight line standing upon another makes the adjacent angles equal, each of these is a right angle.

Thus in Fig. 107 if the line GT standing on EF makes the adjacent angles ETG and FTG equal, then each is a right angle; for each is half of the straight angle ETF.

These angles will be equal if they can be made to coincide, for instance by folding the figure over the line *GT* as crease. Compare the paper folding methods of obtaining a right angle given in § 16.

If two lines form a right angle they are said to be **perpendicular** to each other. Thus in Fig. 107 *GT* is said to be perpendicular to *TF* and to *TE* (or to *EF*).

It is obvious that the halves of equal things are themselves equal. But all straight angles are equal; hence it follows that *all right angles are equal.*

The symbol R will be used in this book to denote a right angle. Thus a straight angle is equal to 2R.

As indicated in § 18 angles are measured in degrees, ninety of which make a right angle. Throughout this book numerical measurements are given in degrees and the decimal of a degree.

Two angles are called **supplementary** if their sum is equal to two right angles, *e.g.* 40° and 140°; or **complementary** if their sum is equal to one right angle, *e.g.* 35° and 55°.

If in Fig. 106 a line rotates about the point *O*, starting from the position *OB* turning in the direction indicated by the arrow, passing through the position *OA*, and returning to the position *OB*, it will then have made a *complete revolution.* But in this operation it will have described two straight angles, viz. one in turning from *OB* to *OA*, and another in turning from *OA* to *OB*.

Thus *in making a complete revolution a line describes two straight angles*, i.e. *four right angles*, or 360°.

51. **Acute, Obtuse, and Re-entrant Angles.**—An angle which is less than a right angle is called **acute**; an angle which is greater than a right angle but less than a straight angle is called **obtuse.** Thus in Fig. 22 (Introd. Course) the first three angles are acute, the last obtuse.

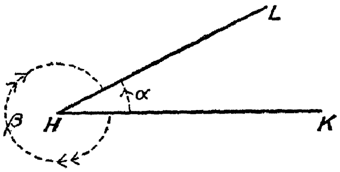

FIG. 108.

An angle which is greater than a straight angle but less than two straight angles is called **re-entrant.**

In Fig. 108 the two lines *HK, HL* are drawn from the point *H*. Now a line in the position *HK* can turn into the position *HL* either by a comparatively small turn in the direction indicated by the single arrow, or by a large turn in the direction indicated by the double arrows. In the first case it describes the angle α ; † in the second case it describes the angle β. The angle β is called the "re-entrant angle *KHL*" to distinguish it from the angle α which is simply called "the angle *KHL*." Thus two lines drawn from a point form two angles, of which the larger is called the re-entrant angle.

52. **Clockwise and Counter-clockwise Rotations.**—If a line is rotating in a plane there are two different directions of turning possible, which are conveniently distinguished by the terms **clockwise** and **counter-clockwise.**

Thus in Fig. 108 the arrow-heads represent the angle β as described by a line turning in the same way as the hands of a clock, *i. e.* by a clockwise rotation ; and the angle α as described counter-clockwise.

53.* **The Definition of an Angle.**—No definition of an angle can be perfectly satisfactory, for the idea is really a fundamental concept (§ 40) implied in the general concept of direction, and an angle merely measures the difference of direction of two lines drawn from a point. But angles so conceived are necessarily less than two right angles, and since angles greater than two right angles enter into our thoughts directly we add together three right angles, for instance, we are bound to adopt a wider basis for the definition of an angle than that of direction. The rotation of a line is suggested, because this is a continual change of its direction, which can be carried on indefinitely.

Euclid's definition—*the inclination of two lines which meet at a point*—is really no definition at all, for the idea of *inclination* or (*slope*) is at least as difficult as that of *angle.*

An angle is not "a figure traced by a revolving line," for this is a sector of a circle (§ 184), nor is it "the space between two intersecting lines," for it cannot be measured in square inches or square centimetres (§ 130) ; nor is it, primarily, "the amount of turning or rotation of a revolving line." It is something described by a straight line when turning. Two angles are equal, *not* when the lines which describe them have turned through equal amounts (for this we have no means of testing), but when one can be exactly fitted on the other (§ 14). Thus angles are not measured by rotations, but rotations are measured by means of angles, just as lines are not measured by distances travelled, but distances travelled are measured by lines.

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† See page xii for the names of the Greek letters used in this book.

* Articles and Exercises *marked with an asterisk* may be omitted in the first reading of the book.

54. Points of the Compass.—Sailors indicate geographical directions by the **points of the compass.** The four *cardinal points,* North, East, South, West, represent directions at right angles; they are shown by the letters *N, E, S, W* on the figure.

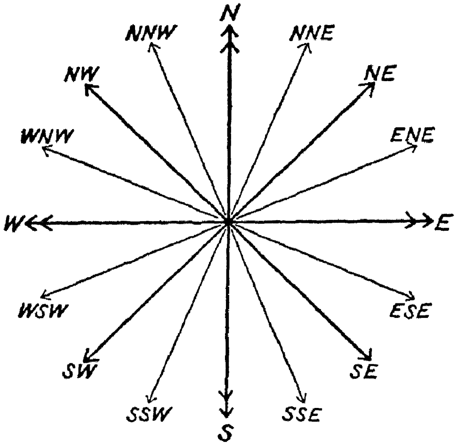

FIG. 109.

The directions which fall midway between the cardinal points are called North-east, South-east, South-west, North-west; they are shown on the figure as *N.E, S.E, S.W, N.W.*

Other intermediate directions are indicated by various combinations of these names, some of which are shown on the figure. These are however of little importance to the student, and are now discarded by sailors for purposes of accurate calculation.

A better method of representing geographical direction is by measuring the angle between the given direction and the nearest cardinal point. Thus in Fig. 110 the direction *OC* is 27° from West; also *OC* is on the south side of West. Thus *OC* is described as 27° *S.* of *W.*†

Example.—Find the angle between two lines which point respectively 32° *W.* of *N.* and 23° *N.* of *E.*

OA and *OB* are the two given lines (Fig. 110).

$\angle AOE = \angle AON + \angle NOE = 32° + 90° = 122°$.

Subtract $\angle BOE$. Thus $\angle AOB = 122° - 23° = 99°$.

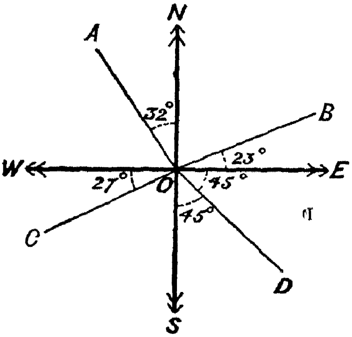

FIG. 110.

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† Sailors usually reckon from North or South, not from East or West Thus *OC* would be 63° *W.* of *S.* Many sailors prefer in complicate calculations to reckon every direction Eastwards from North. Thu *OC* would be 243° *E.* of *N.*

EXERCISES XX. (*Calculations.*)

1. Express in degrees (i) $\frac{1}{12}$ R, (ii) $\frac{3}{40}$ R, (iii) $\frac{7}{50}$ R, (iv) $\frac{17}{60}$ R.

2. A line is revolving in a plane. What angles does it describe in turning through (i) a complete revolution, (ii) one-sixth of a revolution, (iii) two revolutions, (iv) ·4 revolution, (v) 1·4 revolutions, (vi) 2·5 revolutions? Give the answers (*a*) in right angles, (*b*) in degrees.

3. How many times must a line describe an angle of (i) 40°, (ii) 72°, (iii) 120°, in order to complete one revolution?

4. Using Fig. 110 find in degrees the angle described by a line in rotating (i) clockwise from *OD* to *OW*, (ii) counter-clockwise from *OD* to *OW*, (iii) clockwise from *OC* to *ON*, (iv) counter-clockwise from *OC* to *OD*, (v) counter-clockwise from *OS* to *OA*, (vi) counter-clockwise from *OE* to *OC*.

5. Using Fig. 110 find the final position of a line:—(i) which starts from *ON* and turns through 3R in the clockwise direction, (ii) which starts from *ON* and turns through 2·5R in the counter-clockwise direction, (iii) which starts from *OD* and turns through 1·5R in the counter-clockwise direction, (iv) which starts from *OE* and describes an angle of 270° in the clockwise direction, (v) which starts from *OW* and describes an angle of 360° in the counter-clockwise direction, (vi) which starts from *OD* and describes an angle of 180° clockwise, (vii) which starts from *OE* and describes an angle of 8R clockwise, (viii) which starts from *OS* and describes an angle of 11R counter-clockwise.

6. Find the angle between the following pairs of directions:—(i) *N.*, 20° *S.* of *E.* (ii) 20° *W.* of *N.*, 25° *S* of *E.* (iii) 12° *W.* of *S.*, 20° *E.* of *N.* (iv) 24° *W.* of *S.*, 24° *E.* of *N.* (v) 30° *N.* of *E.*, 30° *N.* of *W.*

7. A line points north. If it revolves clockwise through an angle 3R in which direction will it then point?

8. A line points north. If it performs a counter-clockwise rotation of 110° in which direction will it then point?

9 A line points 30° *W.* of *S.* If it performs a clockwise rotation of 130° in which direction will it then point?

10. What is the angle between the hands of a clock at (i) 12 o'clock, (ii) 6 o'clock, (iii) 3 o'clock, (iv) 9 o'clock, (v) 5 o'clock, (vi) 8 o'clock? Give the answers in right angles.

11. What is the angle between the hands of a clock at (i)
three, (ii) half-past eight, (iii) half-past twelve, (iv) half-
Give the answers in degrees.

12. What angle does the minute hand of a clock describe
minutes, (ii) 15 seconds?

55. **Propositions.**—We now enter upon the most important part of our work, the study of Propositions. In learning any proposition it is first necessary that the

student should see the reason for every step in the argument, and the teacher must make certain of this by constant cross-questioning.

Whenever a step depends upon a previous proposition a reference to that proposition is given in the margin.[†] These references must not be reproduced when the proposition is written out for *examination*, but the authors advocate that they should be learnt and used in *class work*. The meaning of any marginal reference can be found from the index at the end of the book. Whether these marginal references are learnt or not, the student must always be able to quote (not necessarily in the words of the book) any previous proposition upon which a step in the argument may depend.

Each proposition must be thoroughly *committed to memory* but *never learnt by heart*. To assist in this the teacher must go over the argument (1) using different lettering, (2) with the figures in a different position, *e. g.* upside down or sideways, (3) mentally, with no visible figure at all.[‡] In private study, the student should adopt a similar course, (1) read the proposition through in the book, (2) repeat it with a figure drawn in a different position and differently lettered, (3) run over it mentally without looking at a figure.

In reproducing the proposition the student need not trouble to use the exact words of the book, nor need he study to avoid them. Whether the marginal references are inserted or not, the proposition must be so worded that the argument is complete without them. The figure need not be the same as in the book so long as it expresses the data.[§] The arguments need not be in the same order so long as the reasoning is sound. The chief purpose of Geometry is to teach sound reasoning.

[†] See for instance the marginal references on p. 93.

[‡] We commend this last method to the attention of teachers. It is possible, even with a middle school class, to go over the whole argument of Euclid I. 47 without a figure, and, of course, in the greater part of this work the figures are much less complicated.

[§] That is the fact or group of facts which the proposition assumes

56. THEOREM [A.1].†—**The two angles which one straight line makes with another straight line on one side of it are together equal to two right angles.**

In Fig. 111 :—

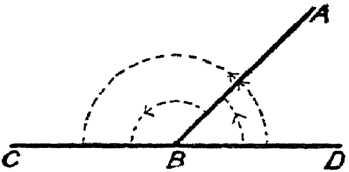

FIG. 111.

Given that CBD is a straight line,
Required to prove that

$\angle ABD + \angle ABC = 2\text{R}.$

Proof.—Suppose a straight line to rotate

(i) from the position BD to the position BA, and

(ii) from the position BA to the position BC.

This line will then describe the two angles ABD, ABC.

But it will rotate from the position BD to the position BC; hence it will describe the straight angle DBC.

Thus $\angle ABD + \angle ABC = \angle DBC.$

Also $\angle DBC = 2\text{R}.$ ‡

Hence § $\angle ABD + \angle ABC = 2\text{R}.$ *Axiom* 1.

COROLLARY 1.—**If one of the angles formed by two intersecting straight lines is a right angle so are all the others.**

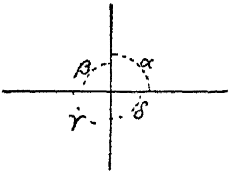

FIG. 112.

For suppose that, in Fig. 112, $\angle\alpha = \text{R}$.

By this proposition $\angle\alpha + \angle\beta = 2\text{R}$, and $\angle\alpha + \angle\delta = 2\text{R}$;

hence $\angle\beta = \text{R}$, and $\angle\delta = \text{R}$.

Also $\angle\beta + \angle\gamma = 2\text{R}$; **hence** $\angle\gamma = \text{R}$.

† This proposition (and any of its corollaries) will be referred to in this book by the abbreviation contained in the square brackets, which stands for "the first proposition on angles."

‡ See the Definition of a Right Angle.

§ The word **Hence** (or its equivalent) in dark type at the beginning of a statement is used in this book to indicate that this statement contains a part of, or the whole of, the required result. In writing out propositions such statements should be doubly underlined.

¶ Full references to Definitions, Axioms, Hypothesis, and Construction are given in the first four propositions only. In later propositions such references are occasionally inserted for the sake of clearness. It is however advisable for the teacher frequently to question the class as to the reason of any step which depends on any of these.

COROLLARY 2.—**If in a plane any number of straight lines are drawn on the same side of a given straight line and from the same point in that line, then the sum of the consecutive angles so formed is equal to two right angles.**

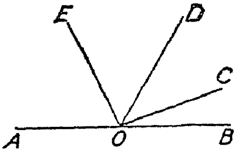

FIG. 113.

For in the accompanying figure the sum of the angles BOC, COD, DOE, EOA is the straight angle BOA, which is equal to two right angles.

COROLLARY 3.—**If in a plane any number of straight lines are drawn from the same point, then the sum of the consecutive angles so formed is equal to four right angles.**

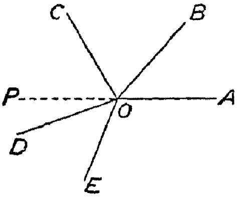

FIG. 114.

For if OA, OB, OC, OD, OE, be the straight lines, and if AO be produced to P, then the sum of the angles AOB, BOC, COD, DOE, EOA is equal to the sum of two straight angles AOP, POA, which is equal to four right angles.

57. Hypothesis and Conclusion.—

DEFINITION.—**In the proof of any theorem a certain fact or group of facts is assumed,** *i.e.* **taken for granted; and a certain fact or group of facts is then proved,** *i.e.* **shown to follow as a consequence. The fact or group of facts which is assumed is called the** *hypothesis* **(or the** *data***); the fact or group of facts which is proved is called the** *conclusion.*

Thus in the preceding theorem (A. 1) the fact assumed is that CBD is a straight line; this is the hypothesis. The fact which is proved is that $\angle ABD + \angle ABC = 2\text{R}$; this is the conclusion.

Note that in this book the hypothesis is always introduced by the words "*Given that*"; and the conclusion by the words "*Required to prove that.*"

58. THEOREM [A.1*c*.]—**If at a point in a straight line, two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in the same straight line.**

In Fig. 115:—*Given that* $\angle ABD + \angle ABC = 2\text{R}$, *Required to prove* that CBD is a straight line.

Cons.†—Produce CB to any point E.‡

Proof.— $\because CBE$ is a straight line, *Cons.*
$\therefore \angle ABE + \angle ABC = 2R.$ A.1.
But $\angle ABD + \angle ABC = 2R.$ *Hyp.*
$\therefore \angle ABE + \angle ABC = \angle ABD + \angle ABC.$ *Axiom* 1.
From each of these equals take $\angle ABC$,
$\therefore \angle ABE = \angle ABD.$ *Axiom* 4.
Hence BE and BD must coincide.
But CBE is a straight line; *Cons.*
hence CBD is a straight line.

59. **Converse Theorems.**—It is obvious that the two theorems which we have now proved are related. *The hypothesis of the first theorem*, viz. that CBD is a straight line, is *the conclusion of the second theorem;* and *the conclusion of the first*, viz. that $\angle ABD + \angle ABC = 2R$, is *the hypothesis of the second.*

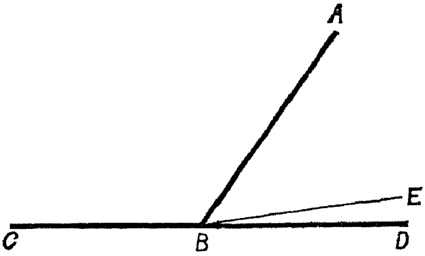

FIG. 115.

When two theorems are related in this way the second is called the **converse** of the first.

The designation of the second theorem, viz. A.1*c.*, is an abbreviation for "the converse of the first proposition on angles."

It is possible for a theorem to be true and its converse untrue. To take a simple illustration, it is true that *if a man is dead he cannot move his arms;* but the converse statement is not true, viz. that *if a man cannot move his arms he is dead.*

Note that the statement *if a man is not dead he can move his arms* is not the converse of the original statement.

60. **Vertically Opposite Angles.**—The two intersecting lines in Fig. 116 form four angles α, β, γ, δ. Of these α, γ are called **vertically opposite**, as they are on opposite sides of their vertex. So also β and δ are vertically opposite.

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† The abbreviations *Cons.* and *Hyp.* will be used for the words *Construction* and *Hypothesis.*

‡ Do not assume either that BE does coincide with BD or that it does not.

THEOREM [A.2]. †—**Each of the angles formed by two intersecting straight lines is equal to the vertically opposite angle.**

In Fig. 116 :—*Given that PQ, ST* are straight lines, *Required to prove* that $\angle \alpha = \angle \gamma$ and $\angle \beta = \angle \delta$.

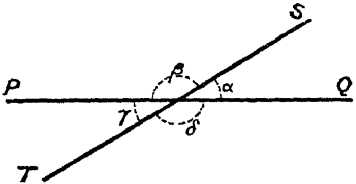

FIG. 116.

Proof.— ∵ *PQ* is a straight line, *Hyp.*

∴ $\angle \alpha + \angle \beta = 2R.$ A.1.

∵ *ST* is a straight line, *Hyp.*

∴ $\angle \gamma + \angle \beta = 2R.$ A.1.

Hence $\angle \alpha + \angle \beta = \angle \gamma + \angle \beta.$ *Axiom* 1.

From each of these equals take $\angle \beta$;

hence $\angle \alpha = \angle \gamma.$ *Axiom* 4.

By an exactly similar argument we can show that

$\angle \alpha + \angle \beta = \angle \alpha + \angle \delta$; **whence** $\angle \beta = \angle \delta.$

EXERCISES XXI.

(*Questions* 1–7 *by Calculation.*)

1. In Fig. 111 if $\angle ABD = 38°$ find $\angle ABC$.

2. In Fig. 111 if $\angle ABD = \frac{1}{3} \angle ABC$, find $\angle ABD$.

3. Four lines *OA, OB, OC, OD* are drawn from a point in the order here given. If $\angle AOB = 30°$, $\angle BOC = 100°$, $\angle COD = 85°$, find $\angle DOA$.

4. If in Fig. 116 $\angle \alpha = 55°$, find the magnitudes of the angles β, γ, δ.

5. If the angle between two lines is 150°, find the magnitude of the re-entrant angle formed by the same two lines.

6. If in Fig. 116 $\angle \alpha = \frac{1}{2} \angle \beta$, find in degrees the magnitude of each of the angles in the figure.

7. If in Fig. 111 $\angle ABD = 40°$, *calculate* the angle between the two lines which bisect the angles *ABC, ABD* respectively.

8. Draw any figure of the type of Fig. 111. Bisect each of the angles *ABC, ABD,* and *measure* the angle between the bisectors.

---

† As an introduction to this theorem, revise Ex. VIII., Ques. 15.

61. **The "Transversal" Diagram.**—Fig. 117 represents a straight line PQ, intersecting two straight lines AB, CD. Eight angles are formed at the two intersections, viz. $\alpha, \beta, \gamma, \delta, \theta, \lambda, \mu, \sigma$. This figure is very important in connection with propositions on parallels.

For convenience of reference—

(i) α, β are called a pair of **adjacent** angles. So also β, γ; μ, σ; σ, θ, etc. Adjacent angles are supplementary.

(ii) α, γ are called a pair of **vertically opposite** angles. So also β, δ; θ, μ; λ, σ.

(iii) $\alpha, \beta, \mu, \sigma$ are called **exterior** angles.

(iv) $\delta, \gamma, \lambda, \theta$, are called **interior** angles.

(v) δ is called the angle **alternate** to λ, and *vice versâ*; γ is called **alternate** to θ, and *vice versâ*.

The line PQ is called a **transversal** (literally a *crossing line*) because it runs *across* the two lines AB, CD.

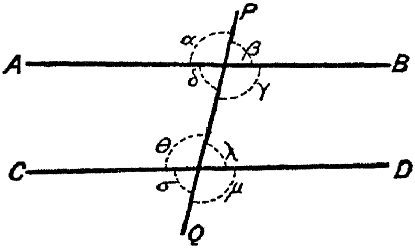

FIG. 117.

We shall use the abbreviation *trv.* for the word *transversal.*

Note that we have proved already [A.1, A.2] that $\alpha=\gamma$, $\beta=\delta$, $\theta=\mu$, $\lambda=\sigma$; $\alpha+\beta=2\text{R}=\gamma+\delta=\lambda+\theta=\mu+\sigma=\alpha+\delta=\beta+\gamma=\theta+\sigma=\lambda+\mu$.

DEFINITION.—**Two straight lines** *in the same plane* **which will not meet, however far they may be produced in either direction, are called parallel.**

Revise § 25. Also Ex. XI., Ques. 4.

62. The next proposition offers considerable difficulty to beginners. The following Preliminary Exercise will afford the student valuable help. (In this exercise the teacher and the class should work together.)

Prel. Ex.—Draw two lines AB, CD and a transversal PQ as in Fig. 118, making each of the angles α, β exactly 65°.

(i) Calculate the size of the angles γ, δ.

(ii) Take an accurate tracing ($bhkd$) of the figure $BHKD$, turn this tracing round (as shown in Fig. 118) and apply it to the figure $CKHA$ so that hk coincides with KH. What do you notice with regard to hb and kd? Can you explain why this happens?

(iii) If HA and KC would meet when produced through A and C, what would you infer about kd and hb, and what about KD and HB? If HA and KC would not meet when produced through A and C, what would you infer? Hence show that AB and CD must be parallel.

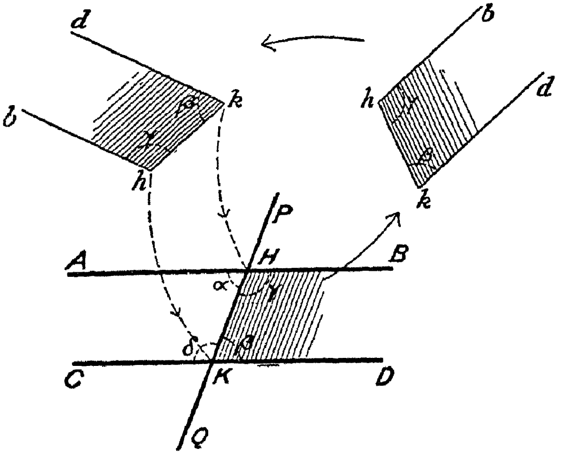

FIG. 118.

THEOREM [P.1].†—**If a straight line meeting two other straight lines in the same plane makes two alternate angles equal to one another, these two straight lines shall be parallel.**

The st. line PQ (Fig. 118) meets the two st. lines AB, CD.

Given that $\angle\alpha = \angle\beta$,

Required to prove that $AB \parallel CD$.

Proof.—(i) *To prove that* $\angle\gamma = \angle\delta$:—

$\because AB$ is a st. line, $\therefore \angle\alpha + \angle\gamma = 2R$. A.1.

$\because CD$ is a st. line, $\therefore \angle\beta + \angle\delta = 2R$. A.1.

Hence $\angle\alpha + \angle\gamma = \angle\beta + \angle\delta$. *Axiom* 1.

From these equals take the equal angles α and β;

thus $\angle\gamma = \angle\delta$. *Axiom* 4.

---

† The abbreviation P.1. denotes the first theorem on Parallels.

(ii) *To show the relation between the figures* BHKD *and* CKHA :—

Suppose that we take an exact duplicate of the figure BHKD † and turn it round as represented in the figure. [See the arrow-heads and shaded diagrams.]

Let the points *b, h, k, d,* be the duplicates respectively of the points *B, H K, D.*

Then the angles at *h, k* are equal to the angles γ, β respectively.

Now apply *bhkd* to the original figure, placing the point *k* on the point *H,* and the line *kh* along the line *HK.* [*Axiom* 10.

$\because$ $kh = HK$, $\therefore$ the point *h* will fall on the point *K.*
$\because$ $\angle\beta$ (at *k*) is equal to $\angle\alpha$ (at *H*), *Hyp.*
$\therefore$ *kd* will lie along *HA.*
$\because$ $\angle\gamma$ (at *h*) is equal to $\angle\delta$ (at *K*) *Proved.*
$\therefore$ *hb* will lie along *KC.*

(iii) *To prove that* $AB \parallel CD$:—

It follows from (ii) that, if *HA* and *KC* meet when produced through *A* and *C,*

then *kd* and *hb* meet when produced through *d* and *b,*

i.e. *KD* and *HB* meet when produced through *D* and *B* ;

but, if *HA* and *KC* do not meet when produced through *A* and *C,*

then *kd* and *hb* do not meet when produced through *d* and *b,*

i. e. *KD* and *HB* do not meet when produced through *D* and *B.*

Thus the lines *AB* and *CD* either meet when produced on both sides of *PQ,* or do not meet on either side.

But the straight lines *AB* and *CD* cannot meet on both sides of *PQ,* for if they have two points in common there can be no space between them. *Def. of st. line.*

Hence *AB* and *CD* do not meet when produced in either direction, *i. e.* they are parallel.

---

† For instance by tracing-paper.

63. THEOREM [P.2].—**If a straight line, meeting two other straight lines, makes an exterior angle equal to the interior and opposite angle on the same side of the line; or if it makes the two interior angles on the same side together equal to two right angles; then the two straight lines shall be parallel.**

The straight line PQ (Fig. 119) meets the two straight lines AB, CD:—

Given that $\angle\alpha = \angle\beta$, (CASE I)

or $\angle\gamma + \angle\beta = 2R$, (CASE II)

Required to prove that $AB \parallel CD$.

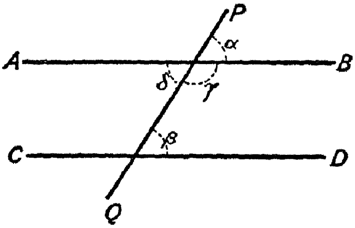

FIG. 119.

Proof.—

CASE I.—Because AB and PQ are intersecting st. lines,

$\therefore \angle\alpha = \angle\delta$. A.2.

Also $\angle\alpha = \angle\beta$. *Hyp.*

Thus $\angle\delta = \angle\beta$.

Hence (using PQ as trv.)† $AB \parallel CD$. P.1.

CASE II.—$\because AB$ is a straight line,

$\therefore \angle\gamma + \angle\delta = 2R$. A.1.

Also $\angle\gamma + \angle\beta = 2R$. *Hyp.*

$\therefore \angle\gamma + \angle\delta = \angle\gamma + \angle\beta$.

From each of these equals take $\angle\gamma$;

thus $\angle\delta = \angle\beta$.

Hence (using PQ as trv.) $AB \parallel CD$. P.1.

---

† When applying any of the theorems P.1, P 2, P.1*c*, or P.2*c*, the student should always mention *the line which is used as the transversal.* The value of this rule will become obvious when it is applied to more complicated figures.

64. It is now necessary to introduce a new axiom known as PLAYFAIR'S AXIOM.

AXIOM (4).—**It is not possible that there should be two different straight lines drawn through the same point and parallel to the same line.**

This statement is an Axiom, for it cannot be formally proved. It is at least in agreement with common experience which represents parallel lines as lines in the same direction, and decides that through a given point there is one and only one line in a given direction.†

Note the close relation between this axiom and Theorem P.3, § 68.

65. THEOREM [P.1c.].—**If a straight line meets two parallel straight lines, it shall make the alternate angles equal to one another.**

The straight line *PQ* (Fig. 120) meets the straight lines *AB, CD.*

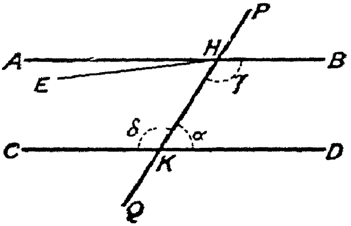

FIG. 120.

Given that $AB \parallel CD$,

Required to prove that $\angle AHK = \angle \alpha$ and $\angle \gamma = \angle \delta$.

Cons.—Let *EH* be a line drawn ‡ through *H* such that *E* and *D* are on opposite sides of *PQ* and $\angle EHK = \angle \alpha$.

---

† It is possible to avoid the use of this axiom, by changing the order in which the propositions are proved, and by assuming (or implying) some other axiom. Euclid used the following axiom :—

If a straight line falling on two other straight lines makes the two interior angles on the same side of it together less than two right angles, then these two straight lines will meet if continually produced on that side on which the angles are together less than two right angles.

The truth of this axiom is not at all obvious. Euclid, however, succeeded in proving the converse statement, viz. that *if the lines meet* on one side of the transversal, then the two angles on that side of the transversal are less than two right angles (see *Euclid*, I. 17). Thus Euclid's axiom practically states that the converse of his 17th proposition cannot be proved but is obviously true.

‡ If there is no such line the whole argument breaks down ; but it is obvious that a trace of the angle α could be applied to the figure with its vertex at *H*, one arm lying along *HK*, and the other arm on the opposite side of *HK* to the point *D* ; this second arm will then indicate the position of the required line *EH*.

Proof.— $\because \angle EHK = \angle \alpha,$ *Cons.*
$\therefore$ (using PQ as trv.) $EH \parallel CD.$ P.1.
Also $AB \parallel CD.$ *Hyp.*

But two different lines cannot be drawn through the same point H parallel to the same line CD;
hence AB and EH are the same line.

Thus $\angle AHK = \angle EHK.$
But $\angle \alpha = \angle EHK.$ *Cons.*
Hence $\angle AHK = \angle \alpha.$

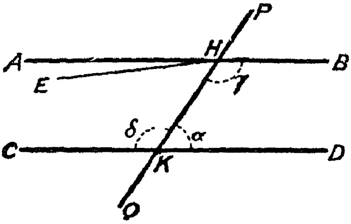

FIG. 120.

Similarly we can prove that :—

If a line FH be drawn through H, such that F and C are on opposite sides of PQ and $\angle FHK = \angle \delta$;
then AB and HF are the same line ; **whence** $\angle \gamma = \angle \delta.$

66. **Hypothetical Constructions.**—In the preceding proposition a "hypothetical construction" was used ; that is to say we supposed that a certain line (viz. EH) was drawn to satisfy a certain condition (viz. to make $\angle EHK = \angle \alpha$) without proving, and without having proved in a previous proposition, that such a line could be drawn. This particular construction was justified in a footnote.

A hypothetical construction should not be used where it can be reasonably avoided, and must never be used unless it can be fully justified. It is illogical to suppose a line or point drawn so as to satisfy certain conditions *unless it is certain that such a line or point exists.*

67. THEOREM [P.2*c*].—**If a straight line meets two parallel straight lines, it shall make each exterior angle equal to the interior and opposite angle on the same side of the line, and each pair of interior angles on the same side of the line together equal to two right angles.**

The st. line PQ (Fig. 121) meets the st. lines AB, CD.
Given that $AB \parallel CD$,
Required to prove that
(i) $\angle \alpha = \angle \theta$, $\angle \beta = \angle \lambda$, $\angle \sigma = \angle \delta$, $\angle \mu = \angle \gamma$; and
(ii) $\angle \theta + \angle \delta = 2\text{R}$, $\angle \gamma + \angle \lambda = 2\text{R}$.

Proof.—(i) Because AB and PQ are two intersecting straight lines,

$\therefore\ \angle\alpha = \angle\gamma.$ A.2.

$\because\ AB \parallel CD,\ \therefore$ (using PQ as trv.) $\angle\theta = \angle\gamma.$ P.1*c*.

Hence $\angle\alpha = \angle\theta.$

Similarly we can show that :— $\angle\beta = \angle\delta = \angle\lambda;$
$\angle\mu = \angle\theta = \angle\gamma;$
$\angle\sigma = \angle\lambda = \angle\delta.$

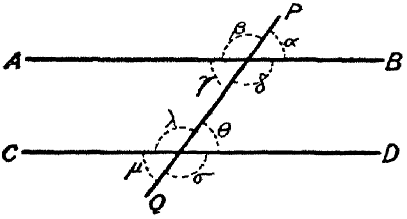

FIG. 121.

(ii) $\because AB \parallel CD,\ \therefore$ (using PQ as trv.) $\angle\theta = \angle\gamma.$ P.1*c*.

To each of these equals add $\angle\delta$;

thus $\angle\theta + \angle\delta = \angle\gamma + \angle\delta.$

$\because\ AB$ is a straight line, $\therefore\ \angle\gamma + \angle\delta = 2R.$ A.1.

Hence $\angle\theta + \angle\delta = 2R.$

Similarly we can show that :—

$$\angle\gamma + \angle\lambda = \angle\theta + \angle\lambda = 2R.$$

68. THEOREM [P. 3].—**If two straight lines are parallel to the same straight line they are parallel to one another.**

In Fig. 122 :—*Given* that
$AB \parallel EF$, and
$CD \parallel EF$,

Required to prove that
$AB \parallel CD.$

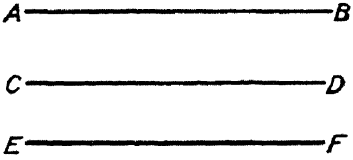

FIG. 122.

Proof.—For if AB and CD are not parallel they will intersect if produced at some point K.

Thus through the point K, two different lines AB, CD have been drawn both parallel to EF.

But *this is impossible* by Playfair's Axiom.

Hence $AB \parallel CD.$

69. The two "Senses" of a Straight Line.—It is obvious that we can travel along a straight line in either of two opposite directions, viz. from A to B (Fig. 123) or from B to A. In some problems it is important to distinguish between these two directions or "**senses**" in a straight line; and we can do this most conveniently by the order of the letters used in naming the line. Thus AB and BA would represent the two opposite senses of the line in Fig. 123; the order of the letters AB indicates that the line is to be regarded as described or drawn from A to B and *vice versâ.*

Similarly PO and PB are lines of *opposite* sense, while BP, OA, PA are all of the *same* sense.

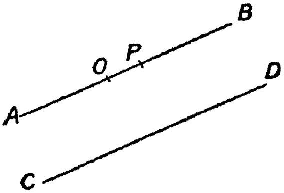

FIG. 123.

Again, if it is required to draw a line from P parallel to DC (Fig. 123), then either of the lines PA or PB may be drawn. But if it is required to draw a line *from* P parallel to and *in the same sense as* DC, then PA is the line required.

Similarly AO and CD are parallel lines taken in the same sense; BP and CD are parallel lines taken in opposite senses.

70. THEOREM [P.4].—**If the two arms of one angle are respectively parallel to the two arms of another, then the angles are either equal or supplementary.**

There are three cases which differ according to the senses in which the lines are drawn.

CASE I.—In Figs. 124 and 125:—

Given that

OA and PC are $\parallel$ lines drawn in the same sense,

and OB and PD are $\parallel$ lines drawn in the same sense;

Required to prove that

$$\angle\alpha = \angle\beta.$$

Cons.—If necessary produce PC to meet OB.

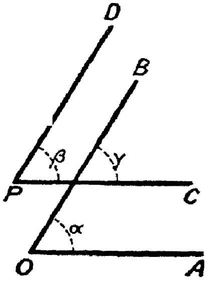

FIG. 124.

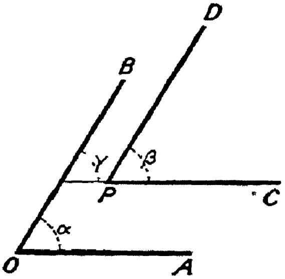

FIG. 125.

Proof.—In either figure,†

$\because OA \parallel PC$, $\therefore$ (using OB as trv.) $\angle\alpha = \angle\gamma$. P.2*c*.

$\because OB \parallel PD$, $\therefore$ (using PC as trv.) $\angle\beta = \angle\gamma$. P.2*c*.

Hence $\angle\alpha = \angle\beta$.

CASE II.—In Fig. 126 :—

Given that

OA and PC are $\parallel$ lines drawn in opposite senses,
and OB and PD are $\parallel$ lines drawn in opposite senses ;

Required to prove that

$\angle\alpha = \angle\beta$.

Cons.—Produce CP and DP to E and F.

Proof.— $\because OA$, PE are $\parallel$ lines drawn in the same sense,

and OB, PF are $\parallel$ lines drawn in the same sense,

$\therefore \angle\alpha = \angle\gamma$. *Case* I.

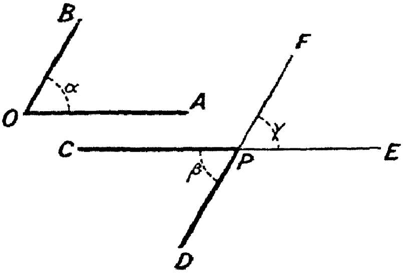

FIG. 126.

$\because CE$ and DF are intersecting straight lines,

$\therefore \angle\beta = \angle\gamma$ A.2.

Hence $\angle\alpha = \angle\beta$.

---

† The validity of the argument depends entirely on the fact that no other type of figure is possible for Case I.

CASE III.—In Fig. 127 :—

Given that

OA and *PC* are || lines drawn in the same sense, and *OB* and *PD* are || lines drawn in opposite senses;

Required to prove that

$$\angle\alpha + \angle\beta = 2R.$$

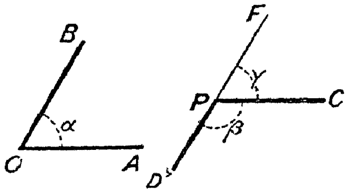

FIG. 127.

Cons.— Produce *DP* to *F*.

Proof.—

∵ *OA* and *PC* are || lines drawn in the same sense,
and *OB* and *PF* are || lines drawn in the same sense,

$\therefore \angle\alpha = \angle\gamma.$ *Case* I.

∵ *DF* is a straight line, $\therefore \angle\gamma + \angle\beta = 2R.$ A. 1.

Hence $\angle\alpha + \angle\beta = 2R.$

Summarizing these results :—

(i) the angles are *equal* if the pairs of parallel arms are either *both* drawn in the same sense from the angular points or *both* drawn in opposite senses; but

(ii) the angles are *supplementary* if *one* pair of parallel arms is drawn in the same sense from the angular points, and the *other* pair in opposite senses.

EXERCISES XXII.

(*Questions* 8 *and* 9 *should be taken as oral questions.*)

1. In Fig. 121, write down all the angles which are equal to $\angle\beta$.

2. In Fig. 121, write down all the angles which are supplementary to $\angle\sigma$.

3. If in Fig. 121 $\angle\alpha = 72°$, write down in degrees the magnitude of each of the other angles.

4. If in Fig. 121 $\angle\alpha = \frac{1}{3}\angle\lambda$, write down the magnitude of angles μ, σ.

5. *PQRS* is a quadrilateral having *PQ* || *SR*. If $\angle P = 100°$, $\angle R = 50°$, calculate $\angle Q$ and $\angle S$. [Use both *PS* and *QR* as transversals.]

6. *ABC* is a triangle and a line *DAE* is drawn through *A* || *BC*, *D* and *B* being on the same side of *AC*. Given that $\angle DAB = 48°$, and $\angle EAC = 62°$, calculate the magnitude of each angle of $\triangle ABC$. [Use both *AB* and *AC* as transversals.]

7. Give a list of all the angles in Fig. 121 which have

(i) both arms parallel to and in the same sense as the arms of the angle γ;

(ii) both arms parallel to and in the opposite sense to the arms of the angle γ;

(iii) both arms parallel to the arms of the angle γ, one arm being in the same sense and the other in the opposite sense.

In each case state whether the angle is equal to or supplementary to γ.

8. Why are the following definitions incorrect?—

(i) If the arms of an angle lie in the same straight line, the angle is called a **straight angle**, or **flat angle**.

(ii) If the arms of an angle are perpendicular, the angle is called a **right angle**.

(iii) Straight lines which do not intersect are called **parallel**.

9. What are the converses of the following statements? Are they true?

(i) If a man is rich he can benefit others.

(ii) If a line is bent, it is not straight.

(iii) If the night is cloudy, the moon cannot be seen.

(iv) If a man has made a "century" in a first-class match, he is a good cricketer.

(v) If two lines meet, they form an angle.

71. **Geometrical Argument.**—The student has now committed to memory a sufficient number of Propositions to understand the nature of the course of logical argument which forms the most important feature of the science of Geometry.

This course of argument deals with geometrical figures which are accurately described by definitions; the definitions are based directly or indirectly on fundamental concepts. The various steps in the argument are obtained by the use of axioms, or axiomatic theorems, or propositions previously proved.

Thus the logical course consists entirely of *fundamental concepts, definitions, axioms, axiomatic theorems, and propositions.* All other matter which this book contains is intended chiefly to explain or illustrate the logical course, and does not form an essential part of this course.

A complete Index to the Definitions and list of the Propositions is given at the end of the book.

In Practical Geometry and in Calculations we use units of length such as the inch and centimetre, and units of angle such as the degree. In the logical course of Theoretical Geometry, however, we avoid (wherever possible) the use of any unit of length or of angle; where a unit of angle cannot be avoided we use the right angle.

It is difficult for the beginner fully to understand the reason for this; but we may state that there are lengths which cannot be measured in inches or centimetres or exact fractions of these units, and angles which cannot be measured in degrees or right angles or exact fractions of these units. Whenever therefore any argument involves the use of units, there is a danger that it may apply only to such quantities as can be measured exactly in terms of certain units, instead of to all possible quantities.

The student must now learn to apply the methods of geometrical argument in proving easy propositions for himself. Geometrical exercises of this type are called **riders**;† they often require considerable ingenuity and form a most valuable mental exercise. In proving riders the student should as far as possible set out his arguments on the model of the propositions proved in the book.

EXERCISES XXIII (Riders).‡

1. If two straight lines are parallel and a transversal is drawn $\perp$ to one of them, it will also be $\perp$ to the other. Use P.2*c*.

2. Prove that straight lines which are $\perp$ to the same straight line are parallel to one another. Use P.2.

3. One straight line standing upon another forms two angles. Prove that the bisectors of these two angles are perpendicular. Use A.1.

4. In a quadrilateral $ABCD$ given that $\angle A + \angle B = 2\text{R}$, prove that $\angle C + \angle D = 2\text{R}$. Use P.2. and P.2*c*.

5. Prove that the opposite angles of a parallelogram are equal to one another. Use P.4.

6. If a quadrilateral have two opposite sides parallel, the sum of its angles is 4R. Use P.2*c*.

7. ABC is a triangle and through A a line DAE is drawn parallel to BC. Use this figure to prove that the three angles of the triangle are together equal to two right angles. Use P.1*c*. and A.1.

8. [A.2*c*.] **If AOB be a straight line and if two other straight lines PO, QO meet it on opposite sides making $\angle POA = \angle QOB$, prove that POQ is a straight line.** Use A.1. and A.1*c*.

---

† In training a class in riders it is sometimes a valuable exercise for the teacher not to use the black-board, but to make each student draw his own figure from a verbal description.

9. [A.2*c.*]† **If four straight lines drawn from a point form four angles each of which is equal to the vertically opposite angle, then shall these four lines form two intersecting straight lines.**
Use A.1 and A.1*c.*

10. AOB is any angle, and OP and OQ are drawn respectively $\perp$ to OA, OB; prove that $\angle POQ$ is either equal or supplementary to $\angle AOB$.
Use *Axiom* 5 and A.1.

11. $\angle AOB$ is any angle, C is any point, and CP and CQ are drawn respectively $\perp$ to OA and OB (produced if necessary); prove that $\angle PCQ$ is either equal or supplementary to $\angle AOB$. Use Ques. 10 and P.4.

12. **Each of the four angles formed by two intersecting lines is bisected. Prove that the bisectors form two perpendicular lines.** (Prove that consecutive bisectors are perpendicular by A.1. Then prove that opposite bisectors are in the same line by A.1*c.*)

13. Prove that there is only one bisector of a given angle. (Prove that if one line is a bisector, no other line can be.)

14. Prove the validity of the construction given in § 25 for drawing a parallel by means of the set square. Use P.2.

15. Prove the validity of the construction given in § 17 for drawing a perpendicular by means of the set square. Use P.2. and P.2*c.*

72. **On Parallelism.**—*Parallel lines drawn in the same sense may be regarded as pointing in the same direction.*

In favour of this we may advance the following arguments:—

(1) The difference of direction between two lines is measured by the angle at their intersection. Now lines which are very nearly parallel form a *very small angle* where they intersect; that is to say they point in *nearly the same* direction. Thus it is natural to regard parallel lines as forming a zero angle, and pointing in the *same* direction.

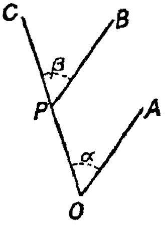

FIG. 128.

(2) If OA and PB are the parallel lines, and OPC a transversal (Fig. 128), then $\angle\alpha = \angle\beta$. (P.2*c.*)

Now the lines OP and PC point *in the same direction*, and the directions of OA and PB differ from the direction OPC by the *equal angles* α and β. Hence it is natural to regard OA and PB as pointing in the same direction.

---

† Riders 8 and 9 and all riders printed in dark type are results which should be remembered by the student. Riders 8 and 9 are lettered "A.2*c.*" (and may be subsequently quoted by these letters) because they are both converses of A.2.

Illustrations.—The following are important illustrations of the principle that parallel lines drawn in the same sense may be regarded as pointing in the same direction.

(*a*) In the plan of a field *all lines pointing due north* are parallel to one another and drawn in the same sense; similarly *all lines pointing due east* are parallel and in the same sense; and so on.

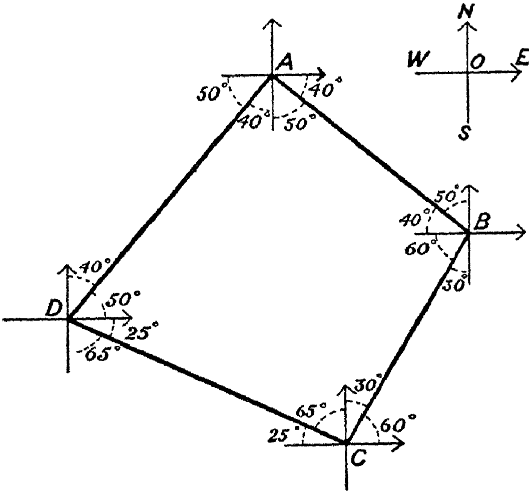

FIG. 129.

Thus, suppose that *A, B, C, D* (Fig. 129) is a plan of a field, and that the thin lines through *O* show the north, south, east, and west directions at *O*; then parallel lines drawn through *A*, *B*, *C*, and *D* show the north, south, east, and west directions at these corners.

Thus, if the various angles are measured, the directions of the sides of the field can now be read off from the figure; *e.g.* *BC* points 30° W. of S., *CB* points 30° E. of N., *CD* points 25° N. of W.; and so on.

(*b*) If a drawing is made on the face of a wall *all vertical lines* on the drawing are parallel to one another.

Suppose that *ABC* (Fig. 130) represents a triangle drawn on the face of a wall, and suppose that a weight *P* hangs by a string from a nail *N*, forming a "plumb line." The direction *NP* is called the *downward vertical*, and the direction *PN* the *upward vertical*.

The directions of the sides of the triangle can then be determined by drawing parallels to *NP*. Thus *AC* points 30° to the right of the downward vertical, *CB* points 80° to the left of the upward vertical, and so on.

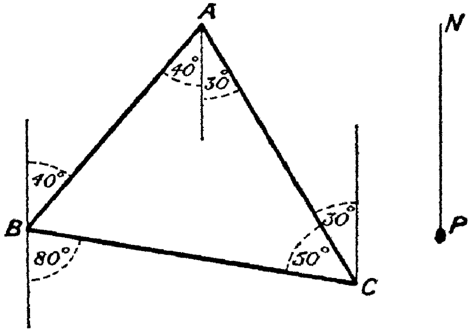

FIG. 130.

Lines drawn at right angles to *NP* are called *horizontal*, and the two senses of these lines may be distinguished as the right horizontal and the left horizontal respectively. Using such lines the directions of the sides of the triangle may be measured from the horizontals instead of from the verticals.

73. **Difference of direction.**—We can now compare the directions of any two lines.

If the two lines are drawn *from* the same point the difference of direction is measured by the angle between them. Thus in Fig. 131 the difference of direction for the two lines *AO* and *OB* (note that the senses are both *from O*) is measured by the angle *AOB*.

If the two lines are drawn from two different points, draw from some one point lines parallel to and in the same sense as each of the given lines; the resulting angle measures the required difference of direction. Thus the difference of direction for the two lines *PC* and *QD* is measured by the angle *AOB*; for *OA*, *OB* are parallel to and in the same sense as *PC*, *QD* respectively.

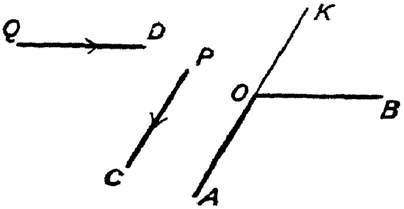

FIG. 131.

Note that the difference of direction for the two lines *AO* and *OB* is *not* the angle *AOB* (for *AO* and *OB* are not both drawn *from O*), but the angle *KOB*. For *OK* is in the same direction as *AO*, and is drawn *from O*.

74. **On Diagrams drawn to scale.**—Every student understands roughly the nature of a *map* of a town, of a *plan* of a field, or a *plan* of the floor of a room; all these are instances of diagrams drawn to scale. For the purpose of using such diagrams note carefully the following points:—

(i) *If we know the* “**scale**” *of this plan we can determine the actual length of any distance which is represented on the plan.*

If a plan of a field is drawn to a scale of $\frac{1}{1000}$, then any distance on the plan is $\frac{1}{1000}$ of the corresponding distance on the field. Suppose that we wish to find the distance between two opposite corners of the field; we measure this distance on the plan (say 3·96″); then the actual distance on the field will be 1000 × 3·96″, *i. e.* 110 yards.

(ii) *Any angle on the plan is* **equal** *to the angle which it represents.*

Thus if one corner of the field is a right angle, it will be represented by a right angle on the plan.

Similarly in drawing a diagram to scale, we make all angles *equal* to the angles which they represent, and all lines *proportional* to the lines which they represent, *i.e.* for the lines we use some fixed scale of representation.

We shall prove in Part II. of this book that it is possible in this way to represent accurately any plane figure by means of a diagram drawn to scale; but for the present the student must assume this without proof and must be content to understand how to use these diagrams. Their properties will not be used in *proving any Propositions,* as it would be logically wrong to base a proof upon any properties which are not regarded as axioms and which have not yet been previously proved.

Example.—There are three towns, A, B, and C. The road from A to B is 14 miles long and its bearing is 20° E. of N.; the road from B to C is 22 miles long and its bearing is 25° N. of E. Find the length and direction of the road from A to C.

We must first choose a convenient scale of representation, say 10 miles to the inch. On this scale a length of 14 miles is represented by 1·4′, and a length of 22 miles by 2·2″.

Take any point A (Fig. 132) to represent the position of the town. Draw AN to represent the north direction at A.

Draw AB 1·4″ long in the direction 20° E. of N.

Draw $Bn \parallel AN$; then Bn represents the north direction at B.

Draw BC 2·2″ long in the direction 25° N. of E., *i.e.* 65° E. of N. The figure now shows the relative positions of the three towns A, B, C.

Join AC; measure AC and $\angle NAC$. AC will be found to measure 3·34″, and therefore represents a distance of 33·4 miles. The angle NAC will be found to measure 47·7°.

Thus the road AC measures 33·4 miles and its bearing is 47·7° E. of N., *i.e.* 42·3° N. of E.

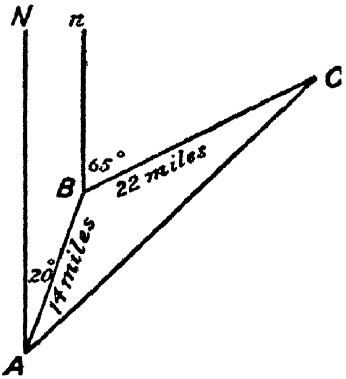

FIG. 132.

EXERCISES XXIV.

(Scale diagrams.)

1. A triangular field has two of its sides each 200 yards long, and the included angle is 73°. Find, by means of a diagram drawn to scale, the length of the third side.

2. If a triangle is drawn having a base of 39 cms. and base angles of 40° and 61° respectively, find the lengths of the sides.

3. Find the distance between two opposite corners of a door which is 3 ft. 8 in. wide and 6 ft. 5 in. high. [Scale 11 in. to 1 cm.]

4. A street is 35 ft. broad. Find by means of a diagram drawn to scale the length of a ladder which will reach from the ground on one side of the street to a window at a height of 46 ft. on the opposite side of the street.

5. A ship sails for 140 miles N.E., and then for 100 miles in a direction 5° N. of W. How far and in what direction must it now sail to return to the starting-point?

6. A triangle PQR is drawn on a vertical wall. PQ is 80 inches long and slopes 18° to the left of the upward vertical; PR is 30 inches long and slopes 50° to the right of the upward vertical. By how many degrees does QR slope from the downward vertical and on which side?

7. $HKLM$ is a four-sided field. KH measures 100 yards and points 30° E. of N.; KL measures 200 yards and points E.; LM measures 200 yards and points 30° E. of N. In what direction does MH point?

8. X and Y are two vessels. At noon Y is 26 miles E. of X. X is sailing due N. at the rate of 3 miles an hour, and Y is steaming 30° E. of N. at the rate of 12 miles an hour. How far is Y from X, and in what direction, at 5 o'clock?

9. A lighthouse-keeper sees a vessel 20 miles away in the direction S.W. How far must the vessel travel due S. before its bearing from the lighthouse is 30° W. of S.?

CHAPTER III.

TRIANGLES.

75. In this chapter we shall discuss the elementary properties of triangles. The student must first revise §§ 11, 12, 23 of the Introductory Course, and then commit to memory the following additional

Definitions :—(1) An **equilateral** triangle is a triangle which has three equal sides.

In Fig. 133 *ABC* is an equilateral triangle.

(2) An **isosceles** triangle is a triangle which has two equal sides.

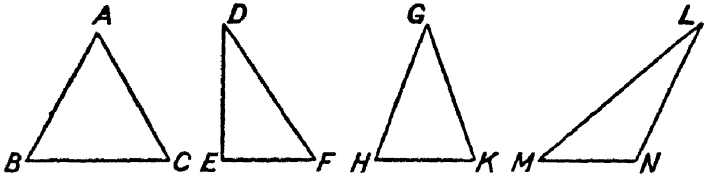

Fig. 133.

In Fig. 133 *GKH* is an isosceles triangle, since the sides *GK* and *GH* are equal. The unequal side *KH* is usually called *the base*, and the cross of the equal sides (*G*) is usually called *the vertex.* In short, we usually regard an isosceles triangle as "standing upon" the unequal side.

(3) A **scalene** triangle is a triangle which has three unequal sides.

In Fig. 133 each of the △s *DEF, LMN* is a scalene triangle.

(4) An **obtuse-angled** triangle is a triangle in which *one* angle is obtuse.

In Fig. 133 *LMN* is an obtuse-angled △, since ∠*N* is obtuse.

(5) A **right-angled** triangle is a triangle in which *one* angle is a right angle.

In Fig. 133 *DEF* is a right-angled △, since ∠*E* is a right angle.

(6) An **acute-angled** triangle is a triangle in which *all three* angles are acute angles.

In Fig. 133 each of the △s *ABC, GKH* is an acute-angled △.

(7) In a right-angled triangle the side opposite to the right angle is called the **hypotenuse**.

In Fig. 133, DF is the hypotenuse of $\triangle DEF$.

76. Revise Ques. 2, 3, 6, 11 in Ex. VIII (Introd. Course).

THEOREM [T.1].†—**If any side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior and opposite angles: also, the three angles of a triangle are together equal to two right angles.**

In $\triangle ABC$, the side BC is produced to D (Fig. 134).

Required to prove that (i) $\angle ACD = \angle A + \angle B$.

(ii) $\angle A + \angle B + \angle \gamma = 2R$.

Cons.—Let CE be $\parallel BA$,‡ and lie on the same side of BD as A.

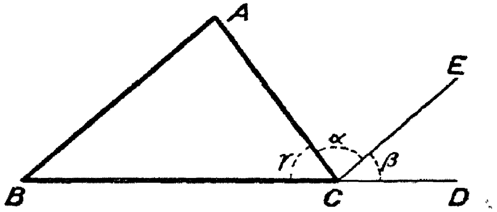

FIG. 134.

Proof.—(i) Since $CE \parallel BA$,

$\therefore$ (using AC as trv.) $\angle \alpha = \angle A$, P.1c.

and (using BD as trv.) $\angle \beta = \angle B$. P.2c.

Adding the equals, $\angle \alpha + \angle \beta = \angle A + \angle B$;

hence $\angle ACD = \angle A + \angle B$.

(ii) To each of these equals add $\angle \gamma$;

thus $\angle ACD + \angle \gamma = \angle A + \angle B + \angle \gamma$.

But since BCD is a straight line,

$\angle ACD + \angle \gamma = 2R$. A.1.

Hence $\angle A + \angle B + \angle \gamma = 2R$.

---

† This abbreviation denotes *the first theorem on triangles.*

‡ Another *Hypothetical Construction* (see p. 92). It is assumed that through any point a line can be drawn parallel to a given line.

COROLLARY 1.—**If one side of a triangle is produced the exterior angle is greater than either of the interior and opposite angles; also any two angles of a triangle are together less than two right angles.**

For, since the exterior angle is equal to the sum of the two interior and opposite angles it is greater than either one of them.

And, since the three interior angles are together equal to two right angles, any two of these angles are together less than two right angles.

COROLLARY 2 [RT.1].†—**In a right-angled triangle the two acute angles are together equal to one right angle.**

For the two acute angles together with the right angle are equal to two right angles.

COROLLARY 3.—**A triangle cannot contain more than one angle which is a right angle or an obtuse angle.**

For otherwise the sum of the three angles could be greater than two right angles.

COROLLARY 4.—**In any quadrilateral the sum of the four angles is together equal to four right angles.**

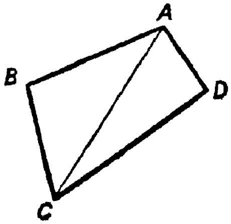

FIG. 135.

For the sum of the four angles of the quadrilateral *ABCD* (Fig. 135) is equal to the sum of the six angles of the two triangles *ABC*, *ADC*. Also the sum of the three angles of each triangle is equal to two right angles.

77*. The following alternative proof of Theorem T.1 (due to John Playfair, 1813) is of great interest. Many mathematicians regard it as preferable to the proof already given; and all are agreed that it is valuable as an illustration of the geometrical concept of direction.

THEOREM [T.1]*.—**If one side of any triangle be produced the exterior angle so formed is equal to the sum of the two interior and opposite angles; also the three angles of any triangle are together equal to two right angles.**

In $\triangle$ *ABC* (Fig. 136), the side *BC* is produced to *D*:—

Required to prove that (i) $\angle\lambda = \angle\alpha + \angle\beta$. (ii) $\angle\alpha + \angle\beta + \angle\gamma = 2R$.

---

† This abbreviation denotes *the first theorem on right-angled* $\triangle$s.

* Articles and Exercises marked with an asterisk may be omitted in a first reading.

(i) *Cons.*—Produce BA, CA, to E, F.

Suppose that a line is moved about on the figure in the manner described below. (The various positions of the line are indicated on the figure by short thick arrows, numbered 1, 2, 3, 4, 5, 6, 7.)†

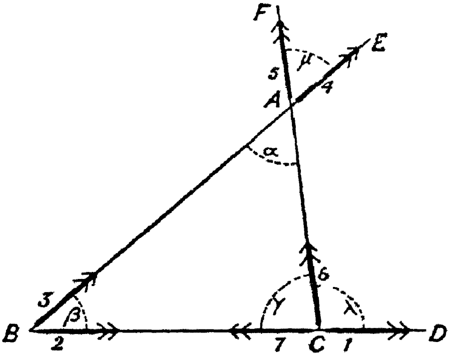

FIG. 136.

The line starts in position 1; it is made to slide back to position 2; it is rotated into position 3; it is made to slide forward into position 4; it is rotated into position 5; it is made to slide back into position 6.

Proof.—In passing from position 1 to position 2, the line does not rotate; *i. e.* its *direction* is unaltered.

In passing to position 3 it rotates through $\angle\beta$.
In passing to position 4 it does not rotate.
In passing to position 5 it rotates through $\angle\mu$.
In passing to position 6 it does not rotate.

Each rotation produces a *change of direction;* moreover, each rotation is counter-clockwise.

Hence the total change of direction in passing from position 1 to position 6 is found by *adding* the two rotations;
i. e. the total change of direction is a counter-clockwise rotation through $\angle\beta + \angle\mu$.

But the total change of direction in passing from position 1 to position 6 can be accomplished by a counter-clockwise rotation (about C) through $\angle\lambda$.

Hence $\angle\lambda = \angle\beta + \angle\mu$.
Because BE, CF are st. lines $\therefore \angle\mu = \angle\alpha$, A.2.
hence $\angle\lambda = \angle\beta + \angle\alpha$.

---

† In each position one end of the line is supposed to be at an angular point of the triangle, but in the figure (for the sake of distinctness) the arrows are slightly separated from the angular points.

(ii) *Cons.*—Suppose the moving line to rotate from the position 6 to the position 7.

Proof.—In passing from position 1 to position 6 the change of direction is a counter-clockwise rotation through $\angle\beta + \angle\alpha$. [*Proved.*

In passing from position 6 to position 7 the change of direction is a counter-clockwise rotation through $\angle\gamma$.

Hence the total change of direction in passing from position 1 to position 7 is a counter-clockwise rotation through $\angle\beta + \angle\alpha + \angle\gamma$.

But the line can pass from position 1 to position 7 by a counter-clockwise rotation (about C) through 2R.

Hence $\angle\beta + \angle\alpha + \angle\gamma = 2\text{R}$.†

78**. **Legendre's Proof.**—A very remarkable proof of Theorem T.1 which is altogether independent of the theory of parallels was given by Legendre. A sketch of this proof is as follows:—

Let ABC be a $\triangle$ *of which BC is the greatest side* (Fig. 137). Bisect AC in D. Join BD and produce it to A_1 so that $DA_1 = BD$. Join A_1C. Prove that $\triangle ADB \equiv \triangle CDA_1$ [A.2, C.1]. Deduce that sum of angles of A_1BC = sum of angles of ABC. Further $BC > BA$, $\therefore BC > CA_1$, therefore $\angle BA_1C > \angle CBA_1$ [T.3],‡ $\therefore \angle ABD > \angle CBD$, $\therefore CBD$ is less than half $\angle B$. Again $\because BC$ is the greatest side of $\triangle ABC$, $\angle A$ is the greatest angle [T. 3], $\therefore \angle A > \angle B$. But $\angle BCA_1 > \angle A$, $\therefore \angle BCA_1$ is the greatest angle in $\triangle BCA_1$ and BA_1 the greatest side. We have now replaced $\triangle ABC$ by $\triangle A_1BC$ with following properties :—

---

† This rotation proof of Theorem T.1 is usually put forward as a proof independent of the theory of parallels; from which the theory of parallels can afterwards be deduced; and it is claimed that this method dispenses with the necessity of any axiom such as Playfair's Axiom.

If, however, we examine this proof it is obvious that we obtain the result by comparing the directions of three lines which do *not* meet in a point, viz. the three sides of the triangle. We are really therefore, in this proof, employing the concept of direction as applied to *all* lines in a plane, and not merely to lines radiating from a point, an extension of the concept which implicitly involves the theory of parallels; and we are also assuming that, in dealing with the directions involved in this concept, we may combine differences of direction by the simple process of addition.

Hence this rotation proof cannot rightly be regarded as independent of the theory of parallels for two reasons:—(i) It is not possible to give any logical account of the concept of direction as applied to all lines in a plane without the use of parallels (see § 73). (ii) It is again necessary to use the theory of parallels in order to prove that differences of direction may be combined by the process of addition.

It is well known that the rotation-proof as above stated applies to spherical triangles, and that the result to which it leads is not true for spherical triangles. This should warn us that some property of the plane, other than is covered by our definition of a plane, is involved in this proof, and since this property cannot be deduced from the definition we are compelled to assume an axiom just as much in this proof as in the other.

** Very difficult.

‡ It must be noted that Theorem T.3 can be proved without using the theory of parallels. See Euclid's order of proof.

(1) The sum of its angles is the same as the sum of the angles of the original triangle.

(2) The two base angles (*i.e.* the two angles adjacent to the greatest side) are together equal to one of the base angles of the original.

(3) $\angle A_1BC$ is less than $\frac{1}{2}\angle ABC$.

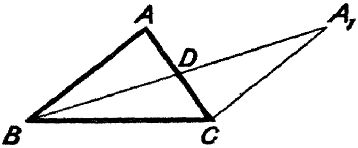

FIG. 137.

Repeat this process continually, at each step using the greatest side as base ; after n repetitions we shall reach a triangle having the same sum for its angles as the original triangle, and having one angle at its base $< B/2^n$. This angle will be the sum of the two base angles of the next triangle of the series, and if n be infinitely great it ultimately becomes infinitely small. The "limit" of the vertical angle is therefore a flat angle, and the sum of the three angles of this, and therefore of every member of the series, is 2R.†

79. DEFINITION.—**A figure which is bounded by more than four straight lines is called a polygon.**

If a polygon has all its sides equal and all its angles equal, it is called a **regular** polygon. The names PENTAGON, HEXAGON, *heptagon*, OCTAGON, *nonagon*, *decagon*, *dodecagon*, *quindecagon*, are used for polygons contained by 5, 6, 7, 8, 9, 10, 12, and 15 sides respectively.

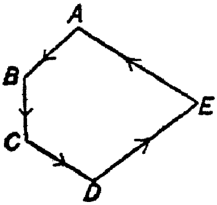

FIG. 138.

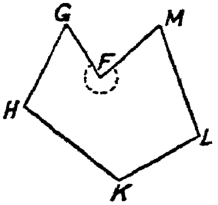

FIG. 139.

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† At first sight it would appear as if in this proof we had a basis for a Theory of Parallels independent of any axiom whatever. For we might now define Parallel lines as lines equally inclined in the same sense to a transversal ; then deduce, easily, by means of T.1, that two lines which were equally inclined to one transversal were equally inclined to every other ; then deduce as in P.1, that two such lines never meet, and thus complete the Theory of Parallels.

Underlying Legendre's proof, there are however axioms essentially of the same nature as Playfair's, and in any case the proof is so difficult that it could not be made, for beginners, the basis of a system of geometrical reasoning. See Cajori's History of Elementary Mathematics, pp. 271–275.

If all the interior angles of a polygon are less than straight angles the polygon is called *convex.* Thus the polygon in Fig. 138 is convex; but the polygon in Fig. 139 is *not* convex, because the interior angle at F is a re-entrant angle.

A point can travel round the sides of a polygon in either of two directions: (i) counter-clockwise as indicated by the arrows in Fig. 138, or (ii) clockwise.

80. *Prel. Ex.*†—In Fig. 140 Of, Og, Oh, Ok, Ol, are respectively parallel to and drawn in the same sense as the sides EA, AB, BC, CD, DE of the pentagon $ABCDE$.

(i) Mark all the pairs of equal angles in the figure. (ii) Prove that the exterior angles of the pentagon together make up 4R. (iii) Prove that the interior angles of the pentagon together make up 6R.

THEOREM [Pn.1].—**If all the sides of a convex polygon be produced in the same direction round the polygon, the exterior angles so formed are together equal to four right angles; and in any convex polygon the sum of the interior angles, together with four right angles, is equal to twice as many right angles as the figure has sides.**

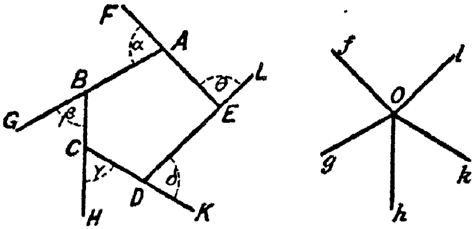


FIG. 140.

(i) In Fig. 140 $ABCD$ . . . is a convex polygon,‡ each of whose sides is produced in the direction indicated by travelling counter-clockwise round the polygon.

Required to prove that $\angle\alpha + \angle\beta + \angle\gamma + \ldots = 4\text{R}$.

Cons.—Let the lines Of, Og, Oh, . . . be drawn from any point O, parallel to and in the same sense as the sides EA, AB, BC, . . . respectively.

---

† These Preliminary Exercises are usually riders which the teacher should take orally with the class as a suitable introduction to the proposition.

‡ The polygon in the figure contains only five sides, but the argument is so worded as to apply to a polygon containing any number of sides.

Proof.—$\because$ Of, Og are drawn parallel to and in the same sense as AF, AB respectively,

$$\therefore \angle \alpha = \angle fOg; \qquad \text{P.4}$$

$\because$ gO, Oh are drawn parallel to and in the same sense as BG, BC respectively,

$$\therefore \angle \beta = \angle gOh. \qquad \text{P.4.}$$

Similarly $\angle \gamma = \angle hOk$, $\angle \delta = \angle kOl$, and so on.

Hence

$$\angle \alpha + \angle \beta + \angle \gamma + \ldots = \angle fOg + \angle gOh + \angle hOk + \ldots$$

Since the angles fOg, gOh, hOk, . . . are the consecutive angles formed at O by the lines Of, Og, Oh, . . .,

$$\therefore \angle fOg + \angle gOh + \angle hOk + \ldots = 4\text{R}. \qquad \text{A.1.}$$

Hence $\angle \alpha + \angle \beta + \angle \gamma + \ldots = 4\text{R}.$

(ii) If the polygon $ABCD$ . . . has n sides,

Required to prove that

sum of all the interior angles of polygon $+ 4\text{R} = 2n\text{R}$.

Proof.—At the angular point A there are two angles, viz. $\angle EAB$ (one of the interior angles of the polygon), and $\angle \alpha$ (one of the exterior angles of the polygon).

Because EAF is a straight line, the sum of these two angles is 2R. A.1.

Similarly at any other angular point there are two angles, one interior and one exterior, whose sum is 2R.

But the polygon has n angular points.

Hence sum of all the interior angles
$+$sum of all the exterior angles $= n \times 2\text{R}$.
$= 2n\text{R}$.

But sum of all the exterior angles $= 4\text{R}$. [Part (i).

Hence sum of all the interior angles $+ 4\text{R} = 2n\text{R}$.

NOTE*.—In this Proposition, Part (i) may be proved by a rotation method, similar to that used in § 77 for Theorem T.1. Suppose a line to start in position EA and to be moved successively into the positions AF, AB, BG, BC, CH, CD, DK, DE, EL, EA. Total change of direction $= \angle \alpha + \angle \beta + \angle \gamma +$ etc. Also total change of direction is equivalent to *one complete revolution*, i.e. to a rotation 4R; for the line has returned to its original position *and sense*. Hence the result follows.

EXERCISES XXV.

CALCULATIONS.

1. Two angles of a triangle measure 50° and 65° respectively. Calculate the third angle.

2. One angle of a triangle measures 46°. If the other two angles are equal, calculate their magnitudes.

3. The three angles of a triangle are equal. Calculate the magnitude of each.

4. Two angles of a triangle are equal, and the third angle exceeds either of them by 30°. Calculate all three angles.

5. Each of the angles at the base of a triangle is double of the vertical angle. Calculate all three angles.

6. Use Theorem Pn.1 to find the sum of the interior angles of a quadrilateral.

7. Find the sum of the interior angles of a polygon of (i) 5, (ii) 6, (iii) 7, (iv) 8 sides.

8. If all the interior angles of a five-sided figure are equal, calculate (i) the exterior angle, (ii) the interior angle, at any angular point.

9. Calculate the magnitude of any interior angle in a regular octagon.

RIDERS.

10. Two angles of one triangle are respectively equal to two angles of a second triangle. Prove that the third angles are also equal. [Use T.1.

11. In a quadrilateral *ABCD*, the sides *AB* and *DC* will meet when produced through *B* and *C*. Prove that the sum of the exterior angles at *B* and *C* is equal to the sum of the interior angles at *A* and *D*. [Use T.1.

12. By dividing a pentagon into three triangles, prove that the sum of its interior angles is equal to six right angles.

13. The two sides of a triangle are produced below the base. Prove that the sum of the two exterior angles is greater than two right angles. [Use A.1.

81. Congruence.—

DEFINITION.—**If two geometrical figures are such that one of them may be made to coincide with the other, line for line and point for point, without any alteration in shape or size, the two figures are said to be congruent.**

Illustration.—If the same figure is printed on two pieces of paper, we obtain two congruent figures. If one of the printings is on tracing paper, we could show this by placing the one figure over the other in such a way that the two figures coincide in all particulars. Similarly in all the copies of this book the triangles *ABC* in Fig. 141 are congruent with one another.

In the above definition note the importance of the words "without alteration in shape or size." The plan of a field has the same shape as the field; but it is not congruent to the field since it is not the *same size.* Again, if we draw a square and a rhombus (Introd. Course, § 26), each figure having each side 1″ long, the two figures are not congruent; for they could not be supposed to coincide without an *alteration in shape.*

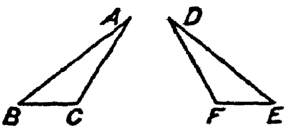


FIG. 141.

To *test* whether two given figures are congruent, take a tracing of the one figure and try to fit it on to the other. It may be necessary to turn the tracing over in order to fit one figure on to the other; for example, the two triangles ABC, DEF (Fig. 141) are congruent, but if we take a trace of $\triangle ABC$ we shall have to turn it over (so that the right side becomes the left) before we can fit the trace on to $\triangle DEF$.

We shall use the symbol $\equiv$ to denote congruence; thus $\triangle ABC \equiv \triangle DEF$
means that $\triangle ABC$ is congruent to $\triangle DEF$.

The process of applying one figure to another to test for congruence is called *superposition.*

EXERCISES XXVI.†

[INTRODUCTORY TO THEOREM C.1.]

Sketch the following triangles without using ruler, scale, or protractor; mark the given lengths and angles on your sketches:—

1. ABC; given, left side $AB = 2''$, right side $AC = 1''$, $\angle A = 50^\circ$.
2. DEF; given, left side $DE = 2''$, right side $DF = 1''$, $\angle D = 50^\circ$.
3. HKL; given, left side $HK = 1''$, right side $HL = 2''$, $\angle H = 50^\circ$.
4. MNO; given, left side $MN = 2''$, right side $MO = 1''$, $\angle M = 40^\circ$.
5. PQR; given, left side $PQ = 1{\cdot}7''$, right side $PR = 1''$, $\angle P = 50^\circ$.

---

† The teacher may prefer to sketch the triangles in Ques. 1–7 on the board, using feet or decimetres instead of inches, and then take Ques. 8–15 orally. Otherwise the examples should be taken partly as an exercise in freehand drawing, and marks given for neatness and approximate accuracy.

6. STV; given, left side $ST = 2''$, right side $SV = 1''$, $\angle T = 50$.

7. XYZ; given, left side $XY = 1''$, base $YZ = 2''$, $\angle Y = 50°$.

If the triangles in Questions 1–7 were all drawn accurately, would you expect the following pairs of triangles to be congruent or not? If not, why not?

8. ABC, DEF. 9. ABC, HKL. 10. ABC, MNO. 11. ABC, PQR.
12. HKL, PQR. 13. ABC, STV. 14. ABC, XYZ.

15. *What test for the congruence of two triangles is suggested by the preceding questions?*

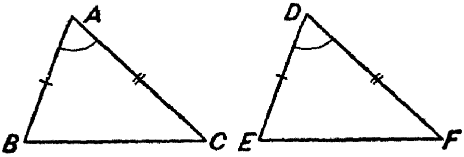


FIG. 142.

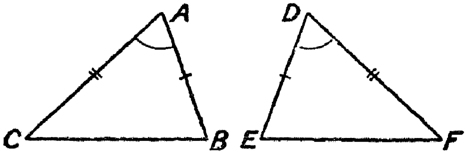


FIG. 143.

82. THEOREM [C.1].—**If two triangles have two sides and the included angle in the one respectively equal to two sides and the included angle in the other, then shall these triangles be congruent.**

In $\triangle$s ABC, DEF (Fig. 142 or 143):—

Given that $\begin{cases} AB = DE, \\ AC = DF, \\ \angle A = \angle D, \end{cases}$

Required to prove that $\triangle ABC \equiv \triangle DEF$, and hence that $BC = EF$, $\angle B = \angle E$, $\angle C = \angle F$.

Cons.—Apply $\triangle ABC$ to $\triangle DEF$, so that
the point A falls on the point D,
the line AB lies along the line DE, and
$\triangle$s ABC, DEF lie on the same side † of AB, DE.

---

† In Fig. 143 the triangle ABC must be reversed. See Axiom 10.

Proof.— $\because$ $AB = DE$, $\therefore$ B falls on E.
$\because$ $\angle A = \angle D$, $\therefore$ AC lies along DF.
$\because$ $AC = DF$, $\therefore$ C falls on F.

$\because$ B falls on E, and C on F,
$\therefore$ the st. lines BC, EF coincide. *Def. of st. line.*

Thus $\triangle$s ABC, DEF coincide in all particulars.

Hence $\triangle ABC \equiv \triangle DEF$;

also $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$.

NOTE 1.—The angles which are given equal must be *included* between the sides which are given equal *in both triangles*, otherwise the triangles are not necessarily congruent.

NOTE 2.—In congruent triangles, *equal angles are opposite to equal sides.* Thus the equal angles B, E are opposite to the equal sides AC, DF respectively; and so on. This rule is often useful when applying the congruence of triangles in later theorems.

EXERCISES XXVII.

[RIDERS ON THEOREM C.1.]

1. ABC is an isosceles triangle, having $AB = AC$. F, E are the middle points of AB, AC respectively. If BE and CF are joined, prove that $BE = CF$ and $\angle ABE = \angle ACF$. [Compare the triangles ABE, ACF.]

2. HKL is an isosceles triangle, having $HK = HL$. The sides HK, HL are produced to M, N; also $KM = LN$. Join KN, LM, and prove that $\angle M = \angle N$. [Compare the triangles HML, HNK.]

3. $PQRS$ is a quadrilateral, having $PS = QR$ and $\angle SPQ = \angle RQP$. Prove that the diagonals PR, QS are equal, and that angles QPR, PQS are equal. [Compare the triangles PQR, QPS.]

4. $ABCDE$ is a regular pentagon. Prove that the lines AC, BD, CE, DA, EB are all equal.

5. ABC is an isosceles triangle having $AB = AC$, and P is any point on the line bisecting $\angle A$. Prove that PBC is an isosceles triangle.

6. $ABCDEF$ is a regular hexagon. Prove that ACE, BDF are both equilateral triangles.

EXERCISES XXVIII.†

[INTRODUCTORY TO THEOREM C.2.]

Sketch the following triangles without the use of ruler, scale, or protractor; mark the given lengths and angles on your sketches:—

1. ABC; base $BC = 2''$, $\angle B = 35°$ (at left extremity of base), $\angle C = 65°$.
2. DEF; base $EF = 2''$, $\angle E = 65°$ (at left extremity of base), $\angle F = 35°$.
3. HKL; left side $HK = 2''$, $\angle H = 35°$ (at vertex), $\angle K = 65°$.

---

† See footnote on Exercises xxvi.

4. MNO ; base $NO = 2''$, $\angle N = 35°$ (at left extremity of base), $\angle M = 65°$.

5. PQR ; base $QR = 2''$, $\angle Q = 35°$ (at left extremity of base), $\angle R = 60°$.

If the triangles in Questions 1–5 were all drawn accurately, would you expect the following pairs of triangles to be congruent or not? If not, why not?

6. ABC, DEF. 7. ABC, HKL. 8. ABC, MNO. 9. ABC, PQR.

10. *What test for the congruence of two triangles is suggested by the preceding questions?*

11. Sketch two triangles STV, XYZ, given that base $TV = 1{\cdot}5''$, $\angle T = 45°$ (at left extremity of base), $\angle S = 55°$, base $YZ = 1{\cdot}5''$, $\angle Y = 45°$ (at left extremity of base), $\angle X = 55°$. If drawn accurately would these two triangles be necessarily congruent?

12. *What test for the congruence of two triangles is suggested by Question* 11?

83. THEOREM [C.2].—**If two triangles have two angles and one side in the one triangle respectively equal to two angles and the corresponding side in the other triangle, then the two triangles shall be congruent.**

CASE I.—When the given angles are adjacent to the given side.

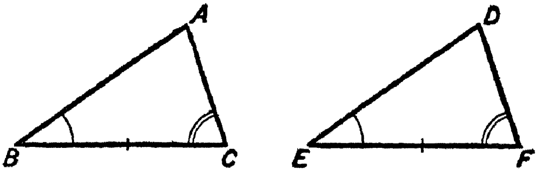


FIG. 144.

In $\triangle$s ABC, DEF (Fig. 144):-

Given that $\begin{cases} \angle B = \angle E, \\ \angle C = \angle F, \\ BC = EF, \end{cases}$

Required to prove that $\triangle ABC \equiv \triangle DEF$;
and hence that $AB = DE$, $AC = DF$, $\angle A = \angle D$.

Cons.—Suppose that $\triangle ABC$ is applied to $\triangle DEF$, in such a way that

B falls on E, BC lies along EF, and
$\triangle$s ABC, DEF lie on the same side of BC, EF.

Proof.—$\because\ BC = EF,\ \therefore\ C$ falls on F.
$\because\ \angle B = \angle E,\ \therefore\ BA$ lies along ED.
$\because\ \angle C = \angle F,\ \therefore\ CA$ lies along FD.
$\because\ BA$ lies along ED, and CA lies along FD,
$\therefore$ the cross of BA and CA falls on the cross of ED and FD,
i.e. A falls on D.

Thus the two triangles coincide in all particulars.

Hence $\triangle ABC \equiv \triangle DEF$,
also $AB = DE,\ AC = DF,\ \angle A = \angle D$.

CASE II.—When the given sides are opposite to one pair of given angles.

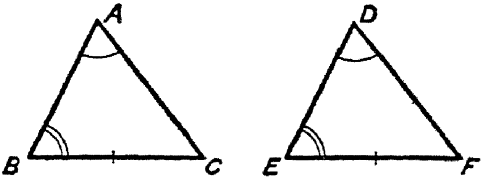


FIG. 145.

In $\triangle$s ABC, DEF (Fig. 145):—

$$\textit{Given that}\quad \left\{\begin{array}{l}\angle A = \angle D,\\ \angle B = \angle E,\\ BC = EF,\end{array}\right.$$

Required to prove that $\triangle ABC \equiv \triangle DEF$;
also $AB = DE,\ AC = DF,\ \angle C = \angle F$.

Proof.—In $\triangle ABC,\ \angle A + \angle B + \angle C = 2\mathrm{R}$; T.1.
also in $\triangle DEF,\ \angle D + \angle E + \angle F = 2\mathrm{R}$. T.1.
$\therefore\ \angle A + \angle B + \angle C = \angle D + \angle E + \angle F$.
But $\angle A + \angle B = \angle D + \angle E$. *Hyp.*
Take the equals from the equals; **thus** $\angle C = \angle F$.

In $\triangle$s ABC, DEF,†

$$\because\ \left\{\begin{array}{l}\angle B = \angle E\\ \angle C = \angle F\\ BC = EF,\end{array}\right.$$

therefore $\triangle ABC \equiv \triangle DEF$;
also $AB = DE,\ AC = DF$. CASE I.

---

† Note this method of setting out the proof of a congruence. The three magnitudes below ABC belong to the triangle ABC, and the three below DEF belong to the triangle DEF.

EXERCISES XXIX.

[RIDERS ON THEOREM C.2.]

1. AD is a line bisecting the angle BAC. From any point D on this line DB, DC are drawn to meet AB, AC and such that $\angle BDA = \angle CDA$. Prove that $DB = DC$.

2. ABC is a triangle, having $AB = AC$. BD is drawn perpendicular to AC, and CE perpendicular to AB. Prove that $BD = CE$.

3. FGH is a triangle, having $FG = FH$. GK is drawn $\perp$ FG to meet FH produced at K, and HL is drawn $\perp$ FH to meet FG produced at L. Prove that $GL = HK$. [First prove that $FL = FK$.]

4. PQR is a triangle, having $\angle PQR = \angle PRQ$. QS is a line bisecting $\angle PQR$ and meeting PR at S; RT is a line bisecting $\angle PRQ$ and meeting PQ at T; ST is joined. Prove that $\triangle QRT \equiv \triangle RQS$, and hence that $\triangle QTS \equiv \triangle RST$. [Use C.1 also.]

84. *Prel. Ex.* (*Pract.*).—Draw any isosceles $\triangle ABC$, having $AB = AC$; bisect $\angle A$ by a line which meets BC at D. Cut the triangle out, and show by folding that the figure is symmetrical[†] about the axis AD.

What do you infer (i) with regard to the $\angle$s B, C, (ii) with regard to $\angle$s ADB, ADC?

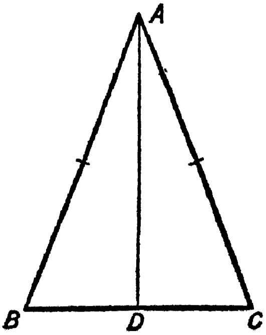


FIG. 146.

THEOREM [T. 2.]—**If two sides of a triangle are equal, then shall the angles opposite to these sides be equal.**

In the triangle ABC (Fig. 146):—

Given that $AB = AC$,

Required to prove that $\angle C = \angle B$.

Cons.—Let the line AD bisect $\angle BAC$,[‡] and let AD meet BC at D.

[†] See Introductory Course, § 28.

[‡] *Hypothetical Construction.* If a line AX revolves from the position AB to the position AC, then $\angle BAX$ increases continuously from the value zero to the value BAC. Hence there must be one position in which $\angle BAX = \frac{1}{2} \angle BAC$, *i.e.* there must be one line which bisects $\angle BAC$.

Proof.—In $\triangle$s ABD, ACD

$$\because \left\{\begin{array}{r} AB = AC, \\ AD = AD, \\ \angle BAD = \angle CAD, \end{array}\right.$$

$$\therefore \quad \triangle ABD \equiv \triangle ACD, \text{ whence } \angle B = \angle C. \qquad \text{C.1.}$$

COROLLARY 1.—**The line which bisects the vertical angle of an isosceles triangle, also bisects the base at right angles.**

$$\begin{array}{l} \because \triangle ADC \equiv \triangle ADB, \\ \quad \therefore DC = DB. \\ \text{also } \angle ADC = \angle ADB, \\ \quad \therefore AD \perp BC. \end{array}$$

COROLLARY 2.—**If a triangle is equilateral it is also equiangular.**

For if also $BC = AB$ (Fig. 146), then $\angle A = \angle C$. Therefore $\angle$s A, B, C are all equal.

85. THEOREM [T.2*c*.]—**If two angles of a triangle are equal then the sides opposite to these angles are also equal.**

In $\triangle ABC$ (Fig. 147):—

Given that $\angle B = \angle C$,

Required to prove that $AC = AB$.

Cons.—Let the line AD bisect $\angle BAC$, and let AD meet BC at the point D.

Proof.—In $\triangle$s ABD, ACD,

$$\because \left\{\begin{array}{r} \angle B = \angle C, \\ \angle BAD = \angle CAD, \\ AD = AD, \end{array}\right.$$

$$\therefore \triangle ABD \equiv \triangle ACD,$$

whence $AB = AC$. C.2.

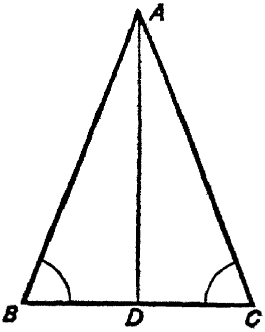


FIG. 147.

COROLLARY.—**If a triangle is equiangular it is also equilateral.**

For if also $\angle A = \angle B$, (Fig. 147) then $BC = AC$. Therefore the sides BC, AC, AB, are all equal.

EXERCISES XXX.

[RIDERS ON THEOREMS T.2 AND T.2*c*.]

1. If the sides of an isosceles triangle are produced below the base prove that the exterior angles so formed are equal.

2. ABC is an isosceles triangle, having $AB=AC$. The angles at B and C are bisected by lines which meet at D. Prove that $DB=DC$.

3. In the preceding question prove that AD bisects $\angle BAC$.

4. In $\triangle PQR$, $PQ=PR$; also a line parallel to QR meets PQ, PR in S, T. Prove that $PS=PT$. [Use T.2; P.2*c*; T.2*c*.]

5. In the figure of Question 4, if QT, RS intersect in O, then OQR, OST are both isosceles triangles. C.1; T.2*c*.

6. In Question 4, prove that $QT=RS$, and $\angle RSP=\angle QTP$. C.1.

86. *Prel. Ex.*—(i) $ABCD$ is a convex quadrilateral in which $AB=BC$, $AD=DC$. Draw the two diagonals and prove (by means of T.2) that $\angle BAD=\angle BCD$. Hence prove that $\triangle ABD \equiv \triangle CBD$.

(ii) $ABCD$ is a quadrilateral having a re-entrant angle at D, in which $AB=BC$, $AD=DC$. Join AC, BD, and prove (by means of T.2) that $\angle BAD=\angle BCD$. Hence prove that $\triangle ABD \equiv \triangle CBD$.

86. THEOREM [C.3].—**If the three sides of one triangle are respectively equal to the three sides of another triangle, the two triangles shall be congruent.**

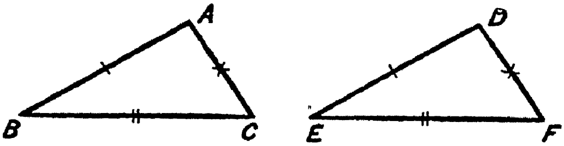


FIG. 148.

In $\triangle$s ABC, DEF (Fig. 148):—

Given that $\begin{cases} AB=DE, \\ BC=EF, \\ CA=FD, \end{cases}$

Required to prove that $\triangle ABC \equiv \triangle DEF$,
and hence that $\angle A=\angle D$, $\angle B=\angle E$, $\angle C=\angle F$.

Cons.—Apply $\triangle ABC$ to $\triangle DEF$, so that
BC coincides with EF (B falling on E), and
A and D fall on opposite sides of EF.
Let $\triangle GEF$ represent the new position of $\triangle ABC$.
Join DG.

CASE I.—When DG cuts EF (Fig. 149).

CASE II.—When DG does not meet EF (Fig. 150).

CASE III.—When DG passes through one extremity of EF (Fig. 151).

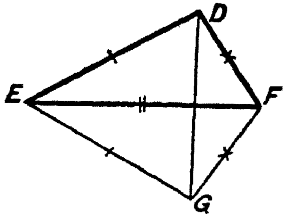


FIG. 149.

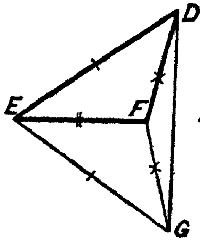


FIG. 150.

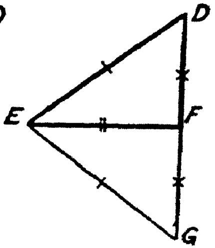


FIG. 151.

Proof.—CASES I. and II.

In $\triangle EDG$, $\because ED = EG$, $\therefore \angle EGD = \angle EDG$. T.2.

In $\triangle FDG$, $\because FD = FG$, $\therefore \angle FGD = \angle FDG$. T.2.

In CASE I. add the equal angles; in CASE II. subtract them. Thus in either case $\angle EGF = \angle EDF$.

CASE III.—In $\triangle EDG$, $\because ED = EG$, $\therefore \angle EGF = \angle EDF$. T.2.

Thus *in all three cases* $\angle EGF = \angle EDF$,

that is (Fig. 148) $\angle A = \angle D$.

In $\triangle$s ABC, DEF,

$$\text{because}\begin{cases} AB = DE, \\ AC = DF, \\ \angle A = \angle D, \end{cases}$$

therefore $\triangle ABC \equiv \triangle DEF$,

also $\angle B = \angle E$, $\angle C = \angle F$. C.1.

EXERCISES XXXI.

[RIDERS ON THEOREM C.3.]

1. In a quadrilateral $ABCD$, given that $AD = BC$, $AC = BD$, prove that $\angle DAB = \angle CBA$, and $\angle ADC = \angle BCD$.

2. In a quadrilateral each side is equal to the opposite side; prove that each angle is equal to the opposite angle.

3. Prove that the diagonals of a rhombus bisect its angles.

4. Prove that the diagonals of a rhombus intersect at right angles. C.3; C.1

87. THEOREM [C.4].†—**If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles shall be congruent.**

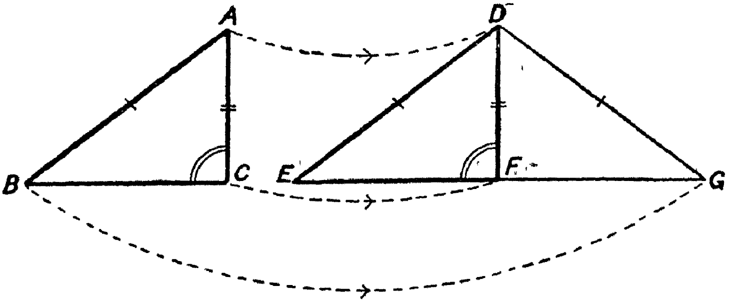


FIG. 152.

In $\triangle$s ABC, DEF (Fig. 152) :—

Given that $\begin{cases} \angle C = \angle F = \mathrm{R}, \\ AB = DE, \\ AC = DF, \end{cases}$

Required to prove that $\triangle ABC \equiv \triangle DEF$.

Cons.—Apply $\triangle ABC$ to $\triangle DEF$, so that
AC coincides with DF (A falling on D), and
B and E fall on opposite sides of DF.
Let $\triangle DGF$ represent the new position of $\triangle ABC$.

Proof.—$\because \angle DFE = \mathrm{R}$, and $\angle DFG$ (*i.e.* $\angle C$) $= \mathrm{R}$,
$\therefore \angle DFE + \angle DFG = 2\mathrm{R}$.

Hence EFG is a straight line. A.1c.

In $\triangle DEG$, $\because DE = DG$, $\therefore \angle G = \angle E$, *i.e.* $\angle B = \angle E$. T.2.

In $\triangle$s ABC, DEF,

$\because \begin{cases} \angle B = \angle E, \\ \angle C = \angle F, \\ AB = DE, \end{cases}$

therefore $\triangle ABC \equiv \triangle DEF$. C.2.

---

† Teachers who prefer to do so may follow this proposition by C.5 (§ 91), and thus complete the theory of the congruence of triangles; but in this case the corollary to C.5 must be deferred.

EXERCISES XXXII.

[INTRODUCTORY TO THEOREM T.3.]

1. In $\triangle$ ABC, if $AB = AC$, what do you know about $\angle$s$\triangle$ B, C?

2. In the same triangle imagine AB lengthened, so that AB is now $> AC$. Would you expect $\angle C$ to be greater or less than before? Would you expect $\angle B$ to be greater or less than before? [Use T.1.]

3. If in $\triangle$ ABC, side $AB >$ side AC, which of the angles B, C would you expect to be the greater?

4. In a scalene triangle, which angle would you expect to be the largest and which the smallest?

88. THEOREM [T.3].—**If one side of a triangle is greater than another, then the angle opposite to the first side shall be greater than the angle opposite to the second.**

In $\triangle$ ABC (Fig. 153):—

Given that $AB > AC$,

Required to prove that $\angle ACB > \angle B$.

Cons.—From AB cut off a part AD equal to AC. Join CD.

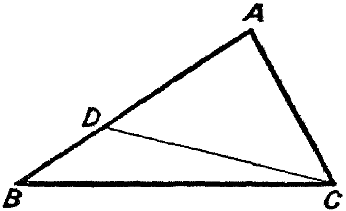


FIG. 153.

Proof.—In $\triangle$ ADC, $\because AD = AC$, $\therefore \angle ACD = \angle ADC$. T.2.

$\because \angle ADC$ is an ext. $\angle$ of $\triangle$ BDC, $\therefore \angle ADC > \angle B$. T.1.

$$\because \left\{\begin{matrix} \angle ACB > \angle ACD \\ \angle ACD = \angle ADC \\ \angle ADC > \angle B \end{matrix}\right\} \text{ therefore } \angle ACB > \angle B.$$

89. THEOREM [T.3c].—**If one angle of a triangle is greater than another, then the side opposite to the first angle shall be greater than the side opposite to the second.**

In $\triangle$ ABC (Fig. 154):—

Given that $\angle B > \angle C$,

Required to prove that $AC > AB$.

Proof.—

AC cannot be less than AB,

for if it were, $\angle B$ would be less than $\angle C$, T.3.

which is not the case.

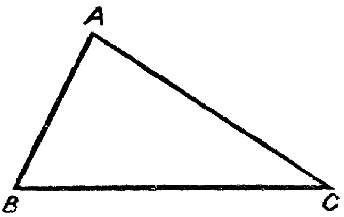


FIG. 154.

AC cannot be equal to AB,
for if it were, $\angle B$ would be equal to $\angle C$, T.2.
which is not the case.
Hence AC must be greater than AB.

90. THEOREM [T.4].—**Any two sides of a triangle are together greater than the third.**

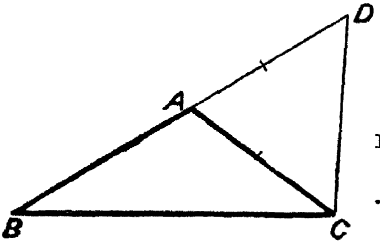


FIG. 155.

In $\triangle ABC$ (Fig. 155):—
Required to prove that
$AB + AC > BC$.

Cons.—Produce BA to D, making $AD = AC$. Join DC.

Proof.—In $\triangle ADC$,
$\because AD = AC, \therefore \angle ACD = \angle D$. T.2.
But $\angle BCD > \angle ACD$;
hence $\angle BCD > \angle D$.

In $\triangle BCD$, $\because \angle BCD > \angle D, \therefore BD > BC$. T.3c.
Now $AC = AD$; to each add AB; thus $AB + AC = BD$.
But $BD > BC$; **hence** $AB + AC > BC$.

EXERCISES XXXIII.

[RIDERS ON THEOREMS T.3, T.3c, and T.4.]

1. The angles at B, C of $\triangle ABC$ are bisected by lines meeting at D. If $AB > AC$, prove that $DB > DC$.

2. In a right-angled triangle the hypotenuse is the greatest side. (T.1.)

3. In a quadrilateral $ABCD$, AB is the least and CD the greatest of the sides. Prove that $\angle A > \angle C$, and $\angle B > \angle D$.

4. The difference of any two sides of a triangle is less than the third side.

5. In an isosceles triangle either of the equal sides is greater than half the base.

6. If ABC be a triangle and D lie in BC, AD is less than half the sum of the three sides.

7. If ABC be a triangle and P any point, then $PA + PB + PC >$ half the sum of the three sides.

8. If $ABCD$ be a quadrilateral and P any point, then the sum of either pair of opposite sides or of the diagonals $< PA + PB + PC + PD$.

91. *Before reading this Theorem revise Prob. IX, Introd. Course, § 23.*

THEOREM [C.5].—**If two sides of one triangle are respectively equal to two sides of another, and if the angles opposite to one pair of equal sides are equal, then shall the angles opposite to the other pair of equal sides be either equal or supplementary.**

In $\triangle$s ABC, DEF (Fig. 156 or 157):—

Given that $\begin{cases} AB = DE, \\ AC = DF, \\ \angle B = \angle E, \end{cases}$

Required to prove that *either* $\angle C = \angle F$, *or* $\angle C + \angle F = 2R$.

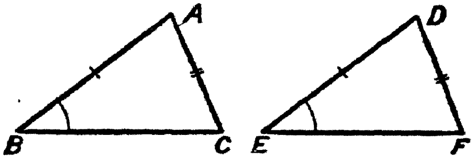


FIG. 156.

CASE I.—When $\angle A = \angle D$ (Fig. 156).

Proof.—

In $\triangle$s ABC, DEF,

$\because \begin{cases} AB = DE, \\ AC = DF, \\ \angle A = \angle D, \end{cases}$

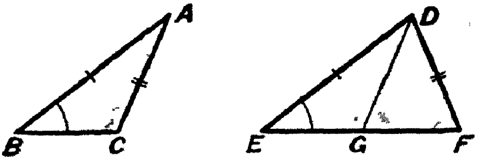


FIG. 157.

therefore $\triangle ABC \equiv \triangle DEF$, and $\angle C = \angle F$. C.1.

CASE II.—When $\angle A \neq \angle D$ (Fig. 157).

Cons.—Since $\angle A \neq \angle D$, one of them must be the greater. Suppose $\angle D > \angle A$; and let a line DG be drawn so that $\angle EDG = \angle A$. Let DG meet EF at G.

Proof.—In $\triangle$s ABC, DEG,

$\because \begin{cases} AB = DE, \\ \angle B = \angle E, \\ \angle A = \angle EDG, \end{cases}$

$\therefore AC = DG$, and $\angle C = \angle DGE$. C.2.

But $AC = DF$, $\therefore DF = DG$;
hence in $\triangle DGF$, $\angle DGF = \angle F$. T.2.

$\because EGF$ is a st. line, $\therefore \angle DGE + \angle DGF = 2R$. A.1.
But $\angle DGE = \angle C$ and $\angle DGF = \angle F$;
hence $\angle C + \angle F = 2R$.

COROLLARY. —**In this Theorem, if in each triangle the given side which is opposite to the given angle is not less than the other given side, the triangles shall be congruent.**

In $\triangle$s ABC, DEF (Fig. 156):—

Given the same facts as above, and *in addition* that AC is not less than AB,

Required to prove that $\triangle ABC \equiv \triangle DEF$.

Proof.—*If possible* suppose that angles C and F are unequal; then they are supplementary (*proved*), and therefore one of them (say C) is obtuse.

But $AC \nless AB$ (*Hyp.*), $\therefore \angle B \nless \angle C$. T.2, and T.3.

Hence $\angle B$ is also obtuse,

i.e. angles B and C are both obtuse.

But *this is impossible.* T.1.

Hence angles C and F are not unequal,

i.e. $\angle C = \angle F$.

In $\triangle$s ABC, DEF,

$$\because \left\{ \begin{array}{l} AB = DE, \\ \angle B = \angle E, \\ \angle C = \angle F, \end{array} \right.$$

therefore $\triangle ABC \equiv \triangle DEF$. C.2.

NOTE 1.—In this Corollary we prove that $\angle C = \angle F$ by showing that any other supposition leads to an impossible result. This method of proof is called *reductio ad absurdum* and is often very useful.

NOTE 2.—Theorem C.4 is a special case of this Corollary; for by T.3*c* in a right-angled triangle the hypotenuse is the greatest side.

92. **Congruent Triangles.**—The following summary of the results which we have now obtained with regard to the congruence of triangles should be carefully studied:—

There are six **elements** in a triangle, viz. three sides and three angles.

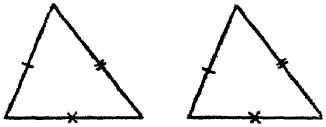
The various tests for the congruence of two triangles all require that *three* elements in the one triangle should be respectively equal to the *corresponding* elements in the other triangle.

Note the importance of the word "corresponding." The three elements in the second triangle must be in the same position *with regard to one another* as the three elements in the first triangle. *Corresponding angles are opposite to corresponding sides* (and are included between pairs of corresponding sides).

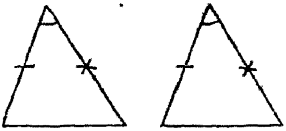
Any set of three elements may be used in the test for congruence excepting (i) three angles and (ii) two sides and the angle opposite to the *shorter* side. Here follows a list of the various cases :—

The triangles are congruent if the following three elements in the one are respectively equal to the corresponding elements in the other :—

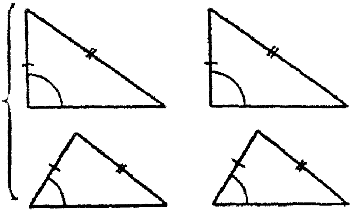
I. Three sides (C.3).



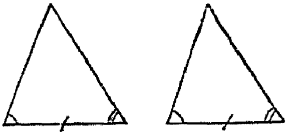
II. Two sides and the included angle (C.1).



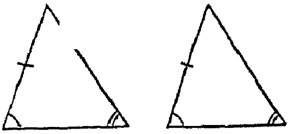
III. Two sides and the angle opposite to one of them, unless this one is shorter than the other (C.4, C.5).



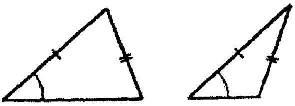
IV. One side and the two adjacent angles (C.2).



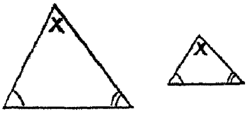
V. One side, the opposite angle, and one adjacent angle (C.2).



The triangles are not necessarily congruent if the following three elements in the one are respectively equal to the corresponding elements in the other :—



VI. Two sides and the angle opposite to the shorter side (C.5).



VII. Three angles.

In Case VII. the triangles will always be the same *shape*, but need not be the same *size*. This case will be treated in Part II.

NOTE.—Compare the various cases of congruence with the various cases of constructing a triangle from three given elements, viz. Problems I, VI, VII, VIII, IX (Introductory Course). Problem IX corresponds to Cases III and VI (above) ; in Case VI the two triangles may not be congruent and accordingly in Problem IX it may be possible to construct two *different* triangles with the given elements.

93. THEOREM [T.5].—**The shortest line which can be drawn from a given point to a given line is the line which is perpendicular to the given line ; and of any other two lines drawn from the given point to the given line on one side of the perpendicular, that which is nearer to the perpendicular is less than that which is more remote.**

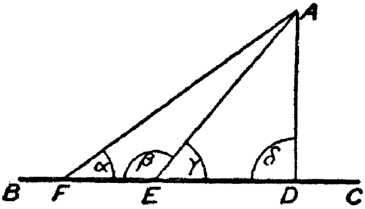


FIG. 158.

In Fig. 158, A is the given point, BC the given line :—

Given that $AD \perp BC$,

Required to prove that
(i) $AD < AE$ or AF,
(ii) $AE < AF$.

Proof.—

(i) In $\triangle ADE$, $\because \angle\delta = \text{R}$,
$\therefore \angle\gamma < \text{R}$. T.1.

In $\triangle ADE$, $\because \angle\delta > \angle\gamma$ (*Proved*), $\therefore AE > AD$; T.3*c*.
that is $AD < AE$.

Similarly $\angle\delta > \angle\alpha$, and **hence** $AD < AF$.

(ii) $\because$ *FED* is a str. line, $\therefore \angle\beta + \angle\gamma = 2R$. A.1.
But $\angle\gamma < R$ (*Proved*); hence $\angle\beta > R$.
In $\triangle AEF$, $\because \angle\beta > R$, $\therefore \angle\alpha < R$. T.1.
In $\triangle AEF$, $\because \angle\beta > \angle\alpha$, $\therefore AF > AE$; that is $AE < AF$. T.3c.

DEFINITION.—**The distance of a given point from a given line is the length of the perpendicular drawn from the given point to the given line (produced if necessary).**

94. *Prel. Ex.*—In Fig. 159, $AC = AG$ and AH bisects $\angle GAC$. Prove that $HC = HG$, and hence that $BC > BG$.

THEOREM [T.6].—**If the two sides of one triangle are respectively equal to the two sides of another, and if the vertical angle of the first triangle is greater than the vertical angle of the second triangle; then the base of the first triangle shall be greater than the base of the second.**

In $\triangle$s *ABC, DEF* (Fig. 159):—
Given that
$$\begin{cases} AB = DE, \\ AC = DF, \\ \angle A > \angle D, \end{cases}$$
Required to prove that
$BC > EF$

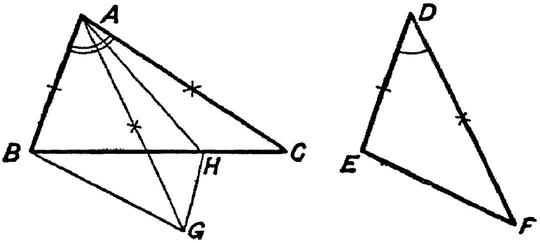


FIG. 159.

Cons.—Apply $\triangle DEF$ to $\triangle ABC$, so that
DE coincides with *AB* (*D* falling on *A*),
and *F* and *C* fall on the same side of *AB*.
Let $\triangle BAG$ represent the new position of $\triangle DEF$.
Let the line *AH* bisect $\angle CAG$, and let *AH* meet *BC* at *H*.
Join *GH*.

Proof.—In $\triangle$s *ACH, AGH*,
$$\because \begin{cases} AC = AG, \\ AH = AH, \\ \angle HAC = \angle HAG, \end{cases}$$
$\therefore \triangle ACH \equiv \triangle AGH$, and $HC = HG$. C.1.

To each add BH; $\therefore BC = BH + HG$.
But, in $\triangle BHG$, $BH + HG > BG$, T.4.
$\therefore BC > BG$, that is $BC > EF$.

95. THEOREM [T.6c].—**If the two sides of one triangle are respectively equal to the two sides of another, and if the base of the first triangle is greater than the base of the second triangle; then the vertical angle of the first triangle shall be greater than the vertical angle of the second.**

In $\triangle$s ABC, DEF (Fig. 160):—

Given that $\begin{cases} AB = DE, \\ AC = DF, \\ BC > EF, \end{cases}$

Required to prove that $\angle A > \angle D$.

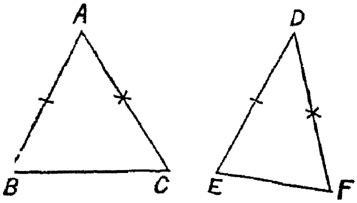


FIG. 160.

Proof.—$\angle A$ *is not equal to* $\angle D$;
for, if it were, then we should have in $\triangle$s ABC, DEF,

$$\begin{cases} AB = DE, \\ AC = DF, \\ \angle A = \angle D, \end{cases}$$

whence BC would be equal to EF; C.1.
which is not the case.

Again $\angle A$ *is not less than* $\angle D$;
for, if it were, then we should have in $\triangle$s ABC, DEF,

$$\begin{cases} AB = DE, \\ AC = DF, \\ \angle A < \angle D, \end{cases}$$

whence BC would be less than EF; T.6.
which is not the case.

$\because$ $\angle A$ is neither equal to nor less than $\angle D$,
therefore $\angle A > \angle D$.

NOTE.—This method of proof is called the *method of exhaustion;* see also § 89.

96. Angles of Elevation and Depression.—

DEFINITION.—**Any line which is parallel to a "plumb-line" is called** *vertical*. **Any line which is at right angles to a vertical† line is called** *horizontal*.

Any body which is falling freely moves vertically downwards.

Any line drawn on a "level" floor is horizontal. Any line lying in the surface of still water is horizontal.

In many problems a horizontal line is a convenient reference line from which to measure directions. In such cases angles measured above the horizontal are called angles of **elevation**, and angles measured below the horizontal are called angles of **depression.**

If a man is looking from a window *W* (Fig. 161) his natural line of sight is the horizontal line *WH*. To look at an object *D* in the street below he must *depress* his line of sight through the angle *HWD*. Hence the angle *HWD* is called the **depression** *of* the point *D at* the point *W*.

To look at an object *E* at a higher level than the window he must *elevate* his line of sight through the angle *HWE*. Hence the angle *HWE* is called the **elevation** *of* the point *E at* the point *W*.

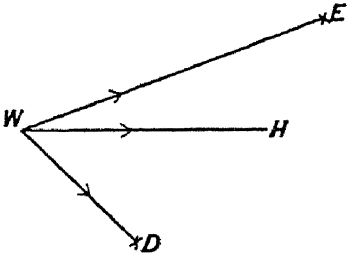


FIG. 161.

Example.—A man is standing at the top of a cliff 150 ft. high. If the angle of depression of a boat on the sea is 25°, determine the distance of the boat from the foot of the cliff.

A convenient scale of representation will be 100 ft. to 1 inch.

Let *A* represent the foot of the cliff, and *AX* the surface of the sea (Fig. 162).

A height of 150 ft. will be represented by 1·5″; draw *AB* ⊥ *AX* of length 1·5″. Then *B* represents the top of the cliff.

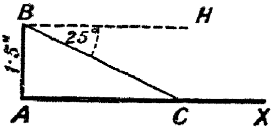


FIG. 162.

Draw *BH* ∥ *AX*; then *BH* represents the horizontal through *B*.

Draw a line *BC*, making ∠*HBC* = 25°, meeting *AX* at *C*. Then *C* represents the position of the boat.

By measuring we find *AC* = 3·22″; hence *AC* represents a distance of 3·22 × 100, or **322 ft.**

NOTE.—In solving this problem squared paper can be used with advantage. *AB*, *AX*, and *BH* will then lie along ruled lines.

EXERCISES XXXIV.

CALCULATIONS.

1. Find the magnitude of each acute angle in a right-angled isosceles triangle.

2. The angles at the base of a triangle are 35° and 65° respectively. If the vertical angle is bisected calculate the angles which the bisector makes with the base.

3. In the preceding triangle a line is drawn from the vertex perpendicular to the base. Calculate the angles between this line and the sides of the original triangle.

4. If two angles of a triangle are together equal to the third, calculate the third.

5. The base angles of a triangle are 30° and 45° respectively. Two lines drawn from the vertex to the base divide the vertical angle into three equal parts. Calculate the angles of the triangle formed by these two lines and the base.

6. The two sides of a triangle are produced below the base. If the two exterior angles so formed measure 120° and 85° respectively, calculate the vertical angle of the triangle.

7. The interior angles of a six-sided figure are all equal. Calculate their magnitude.

8. In a quadrilateral three angles are respectively twice, three times, and four times as great as the fourth angle. Calculate the angles.

9. Each of the exterior angles of a regular polygon is 45°. Calculate the number of sides in the polygon.

10. The exterior angles of a polygon are alternately 60° and 30°. Find the number of sides.

11. If a line points at an angle of 20° below the horizontal, find the angle which it makes with the upward vertical.

12. A line makes an angle of 115° with the downward vertical; find its angle of elevation or depression.

13. A line originally points vertically up. Find its angle of elevation or depression after rotating through an angle of 800°.

SCALE DIAGRAMS.

14. The angle of elevation of the top of a vertical pole at a point on the ground 30 ft. away from its foot is 38°. Find the height of the pole.

15. A lighthouse is 70 ft. high, and the angle of depression of a floating buoy when seen from the top of the lighthouse is 15°. Find the distance of the buoy from the top of the lighthouse.

16. Find the elevation of the top of a church-spire 100 ft. high at point on the ground 350 ft. from the church.

17. At a distance of 100 ft. from the wall of a house the elevation of the top of the wall is 25°. Find the length of a ladder which will just reach to the top of the house from a point on the ground 25 ft. from the house.

18. The elevation of the top of a high rock at a certain point is 20° ; the observer walks on level ground 50 yards toward the rock, and then finds that the elevation is 45°. Find the height of the rock.

19. At a distance of 50 ft. from a flagstaff the elevation of the top is 32°. How far must the observer walk toward the flagstaff before the elevation of the top is 55° ?

20. Two large stones are placed on the ground at a distance of 50 ft., each lying due N. of a tower. Find the height of the tower if the angles of depression of the stones at the top of the tower are 30° and 55° respectively.

21. A balloon is seen from two stations at the same moment. At the one station, due east of the balloon, its elevation is 50° ; at the other station due west of the balloon its elevation is 22° ; the stations are one mile apart. Find in yards the actual distance of the balloon from the first station.

RIDERS.

(*In Questions* 22–36 *the references indicate* all *the propositions required in the proof.*)

22. The line AB is bisected at right angles by the line CD. Prove that any point on CD is equidistant from the points A and B. C.1.

23. The angle ABC is bisected by the line BD. Prove that any point on BD is equidistant from the lines AB and BC. C.2.

24. The point P lies within the angle ABC and is equidistant from AB and BC. Prove that PB bisects the angle ABC. C.4.

25. [T.2.] **The join of the vertex of an isosceles triangle and the middle of its base bisects the vertical angle.** C.3.

26. [T.2.] **The line from the vertex of an isosceles triangle perpendicular to the base bisects the base and the vertical angle.** C.4.

27. Prove that, if a straight line is drawn from an angular point of an equilateral triangle to any point in the opposite side, then this line is longer than either of the parts into which the side is divided, but shorter than their sum. T.1, T.2, T.3*c*.

28. ABC is an isosceles triangle, having the side AB equal to the side AC ; the side BA is produced to D, and the exterior angle CAD is bisected by the straight line AX. Prove that AX is parallel to BC.
P.1, T.1, T.2.

29. If the straight line which bisects an exterior angle of a triangle is parallel to the base of the triangle, prove that the triangle is isosceles. P.1*c*, P.2*c*, T.2*c*.

30. The bisectors of the angles at the base of a triangle meet at a point within the triangle ; prove that this point is equidistant from the three sides of the triangle.

31. If two circles intersect, the join of their centres bisects at right angles the straight line joining their points of intersection. C.1, C.3.

32 In the triangle ABC the angle ABC is bisected by the straight line BD meeting AC in D. Prove that AB is greater than AD and CB than CD. T.1, T.3*c*.

33. ABC is an acute-angled triangle, and O is a point within it, such that OA, OB, and OC are all equal. Prove that the angle BOC is double the angle BAC. Produce AO : T.1, T.2.

34. In a right-angled triangle if one acute angle be double the other acute angle, the hypotenuse will be double one of the sides. (Show that the figure is half of an equilateral triangle.) A.1, C.1, T.2*c*.

35. If O is any point within a triangle ABC, prove that angle BOC is greater than angle BAC. [Produce BO to meet AC; use T.1 twice.]

36. In the figure of Question 35, prove that $OB+OC$ is less than $AB+AC$. [Produce BO to meet AC; use T.4 twice.]

In Questions 37–47 the references indicate some of the propositions required in the proof.

37. In the triangle ABC join the vertex A to M the middle point of the base BC, and prove that the angle AMB is greater than, equal to, or less than the angle AMC according as AB is greater than, equal to, or less than AC. T.6*c*.

38. A and B are two points on the same side of the straight line CD. From A draw AP perpendicular to CD, and produce AP to E, making PE equal to AP. Join BE, cutting CD in X; join AX. Prove that the angle AXC is equal to the angle BXD. A.2.

39. ABC is any triangle and the angles at B and C are bisected by the straight lines BI and CI, meeting at I within the triangle ABC; through I a straight line PIQ is drawn parallel to BC and meeting the sides AB and AC in P and Q respectively; prove that BP and CQ are together equal to PQ. P.1*c*.

40. ABC is an isosceles triangle; DEF is a straight line perpendicular to the base BC, meeting AB in E and CA produced in F; show that the triangle AEF is isosceles. T.1.

41. Given a convex polygon, join each angular point to a point within the polygon. Use this construction to prove that the sum of the interior angles of the polygon is equal to $(2n-4)$ right angles. T.1.

42. In the triangle ABC, $AB=AC$. Prove that $\angle C=\angle B$, *by comparing the triangles* ABC, ACB. C.1.

43. In the triangle ABC, $\angle B=\angle C$. Prove that $AC=AB$, *by comparing the triangles* ABC, ACB. C.2.

44. AB is a given line and C is a given point without AB. Prove that it is not possible to draw three equal straight lines from C to AB. T.5.

45. If the base BC of a triangle ABC be produced to D, the angle between the bisectors of the angles ABC, ACD is equal to half the angle BAC. T.1.

46. If the sides *AB, AC* of a triangle *ABC* be produced to *D* and *E*, and the bisectors of the angles *CBD, BCE* meet in *I*, prove that the angle *BIC* is equal to half the sum of the angles *ABC, ACB*. A.1, T.1.

47. Show how to divide a right-angled triangle into two isosceles triangles, and hence prove that the straight line drawn from the middle point of the hypotenuse to the right angle is equal to half the hypotenuse. T.1.

48. If two quadrilaterals have three sides of the one equal to three sides of the other, each to each, and have also the angles contained by those sides equal, each to each, then the quadrilaterals shall be equal in all respects.

49. If two convex quadrilaterals have the four sides of the one respectively equal to the corresponding sides of the other, and also one angle in the one equal to the corresponding angle in the other, the quadrilaterals are congruent.

50. *ABC* is a triangle, and the sides *AB* and *AC* are produced to *D* and *E*. If the angle *DBC* is equal to the angle *ECB*, prove that *AB* is equal to *AC*.

51. *ABC* is an isosceles triangle having the side *AB* equal to the side *AC*. Bisect the angles *ABC, ACB* by the straight lines *BY, CZ* which meet *AC* and *AB* in *Y* and *Z* respectively. Prove that the triangles *YBC, ZCB* are equal in all respects.

52. Prove that the triangle formed by joining the three middle points of the three sides of an isosceles triangle is isosceles.

53. In a right-angled triangle a line is drawn from the vertex of the right angle perpendicular to the opposite side. Prove that each of the smaller triangles into which the original triangle is divided is equiangular to the original triangle.

54. *A* is the vertex of an isosceles triangle *ABC*, and *BA*, one of the equal sides, is produced to *D*, so that *AD* is equal to *AB* or *AC*; join *DC* and show that *DC* is at right angles to *BC*.

55. Prove that if the exterior angles *DBC, ECB* of the triangle *ABC*, made by producing *AB* and *AC* to *D* and *E*, be bisected by the straight lines *BO* and *CO* meeting in *O*, then *O* is equidistant from *AD, BC, AE*.

56. The straight line which joins the vertices of two isosceles triangles drawn on opposite sides of the same base, is perpendicular to the base.

57*. A circle cannot cut a straight line in more than two points.

58*. In Fig. 117 given that $\angle\gamma + \angle\lambda < 2\text{R}$, prove that *AB* and *CD* meet when produced through *B* and *D*.

[Prove (i) that *AB* is not parallel to *CD*; (ii) that *AB* and *CD* do not meet when produced through *A* and *C*.]

59*. In the same figure, given that $\angle\gamma + \angle\lambda > 2\text{R}$, prove that *AB* and *CD* meet when produced through *A* and *C*.

[Prove (i) that *AB* is not parallel to *CD*; (ii) that *AB* and *CD* do not meet when produced through *B* and *D*.]

60*. Prove that the three straight lines which bisect the three angles of an equilateral triangle meet in a point.

CHAPTER IV.

PARALLELOGRAMS.

97. LEARN the definitions of **Parallelogram, Rectangle,**† **Rhombus,** and **Square** given in § 26.

The symbol $\|^m$ will be used for the word parallelogram. The abbreviations *rect.*, *rhom.*, and *sq.* will be used for the words rectangle, rhombus, and square respectively.

Revise Exercises XII, p. 46.

THEOREM [Pm.1].—**In any parallelogram, opposite sides are equal, and opposite angles are equal; also each diagonal divides the parallelogram into two congruent triangles.**

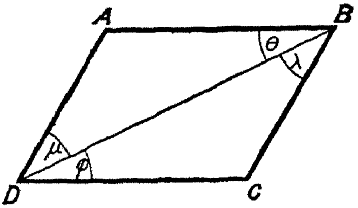


FIG. 163.

Given that $ABCD$ is a $\|^m$ (Fig. 163),
Required to prove that
$AB = DC$, $AD = BC$;
$\angle A = \angle C$, $\angle ABC = \angle ADC$;
$\triangle ABD \equiv \triangle CDB$,
$\triangle ABC \equiv \triangle CDA$.

Proof.—

$\because AB \parallel DC$ (*Def.*), $\therefore$ (using BD as trv.) $\angle\theta = \angle\phi$; P.1*c*.

$\because AD \parallel BC$ (*Def.*), $\therefore$ (using BD as trv.) $\angle\lambda = \angle\mu$. P.1*c*.

Add these equals; **thus** $\angle ABC = \angle ADC$

$$\text{In } \triangle\text{s } ABD,\ CDB,\ \because \left\{\begin{array}{l} \angle\theta = \angle\phi \\ \angle\mu = \angle\lambda \\ BD = DB \end{array}\right\}$$

therefore $\triangle ABD \equiv \triangle CDB$; C.2.

hence $AB = DC$, $AD = BC$, and $\angle A = \angle C$.

Similarly, if AC be joined, it can be proved that
$\triangle ABC \equiv \triangle CDA$.

---

† Strictly speaking a rectangle should be defined as a quadrilateral in which three angles are right angles. Then, since the four angles are together equal to four right angles (T.1), the fourth angle of the rectangle must also be a right angle. It will be proved in § 98 that a rectangle is a parallelogram.

98. THEOREM [Pm.1c].—(i) **If the opposite sides of a quadrilateral are equal, it is a parallelogram; also,** (ii) **if the opposite angles of a quadrilateral are equal, it is a parallelogram.**

(i) In the quadrilateral $ABCD$ (Fig. 163):—
Given that $AB = DC$, and $AD = BC$,
Required to prove that $ABCD$ is a $\|^{m}$.

Cons.—Join BD.

Proof.—In $\triangle$s ABD, CDB,

$$\because \begin{cases} AB = CD \\ AD = CB \\ BD = DB \end{cases} \therefore \triangle ABD \equiv \triangle CDB; \qquad \text{C.3.}$$

$$i.e. \ \angle\theta = \angle\phi, \ \angle\mu = \angle\lambda.$$

$\because \angle\theta = \angle\phi$, $\therefore$ (using BD as trv.) $AB \parallel CD$; P.1
and $\because \angle\mu = \angle\lambda$, $\therefore$ (using BD as trv.) $AD \parallel BC$. P.1.
$\because AB \parallel CD$ and $AD \parallel BC$, **therefore** $ABCD$ is a $\|^{m}$. *Def.*

(ii) In the quadrilateral $ABCD$ (Fig. 163):—
Given that $\angle A = \angle C$, and $\angle B = \angle D$,
Required to prove that $ABCD$ is a $\|^{m}$.

Proof.—$\because \angle A = \angle C$, and $\angle B = \angle D$,

$$\therefore \angle A + \angle B + \angle C + \angle D = 2\angle A + 2\angle B.$$

But, in quad. $ABCD$, $\angle A + \angle B + \angle C + \angle D = 4\text{R}$. T.1.

$$\therefore \ 2\angle A + 2\angle B = 4\text{R},$$

$$i.e. \ \angle A + \angle B = 2\text{R};$$

$\therefore$ (using AB as trv.) $AD \parallel BC$.

Similarly it can be proved that

$$\angle A + \angle D = 2\text{R},$$

$$\text{and } \therefore AB \parallel DC.$$

$\because AD \parallel BC$, and $AB \parallel DC$, **therefore** $ABCD$ is a $\|^{m}$. *Def.*

COROLLARY 1.—**A rhombus is a parallelogram; and a square is a parallelogram.**

For in either a rhombus or a square opposite sides are equal.

COROLLARY 2.—**A rectangle is a parallelogram.**

For in a rectangle opposite angles are equal.

99. THEOREM [Pm.2].—**If in a quadrilateral two opposite sides are equal and parallel, then the quadrilateral is a parallelogram, and the other two sides are also equal and parallel.**

Given that $AB = DC$, and $AB \parallel DC$, (Fig. 163),
Required to prove that
$ABCD$ is a $\parallel^m$, $AD = BC$, and $AD \parallel BC$.
Cons.—Join BD.

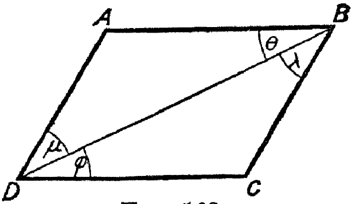


FIG. 163.

Proof.—$\because AB \parallel DC$, $\therefore$ (using BD as trv.) $\angle\theta = \angle\phi$. P.1*c*.

In $\triangle$s ABD, CDB,

$$\because \left\{\begin{array}{l} AB = CD \\ BD = DB \\ \angle\theta = \angle\phi \end{array}\right\}$$

therefore $AD = BC$ and $\angle\mu = \angle\lambda$. C.1.

$\because \angle\mu = \angle\lambda$, $\therefore$ (using BD as trv.) $AD \parallel BC$. P.1.

$\because AB \parallel DC$, and $AD \parallel BC$, therefore $ABCD$ is a $\parallel^m$.

100. THEOREM [Pm,3].—**If one angle of a parallelogram is a right angle, so are all the others.**

Given that $ABCD$ is a $\parallel^m$ (Fig. 164),
and $\angle A = R$,

Required to prove that

$$\angle B = \angle C = \angle D = R.$$

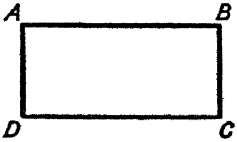


FIG. 164

Proof.—$\because AD \parallel BC$, $\therefore$ (using AB as trv.) $\angle A + \angle B = 2R$. P.2*c*.

But $\angle A = R$; hence $\angle B = R$.

In $\parallel^m ABCD$, $\angle C = \angle A$,
and $\angle D = \angle B$. Pm.1.

Hence $\angle C = \angle D = R$.

COROLLARY.—**All the angles of a square are right angles.**

For by Pm.1*c*. [*Cor.* 1], a square is a parallelogram.

EXERCISES XXXV (*Riders*).

1. If the opposite angles of a quadrilateral are equal, the opposite sides are also equal.

2. Prove that any straight line drawn through the middle point M of the diagonal of a parallelogram and terminated by opposite sides of the parallelogram is bisected at M.

3. The diagonals of a square divide it into four congruent triangles.

4. If one angle of one parallelogram is equal to one angle of another parallelogram, the remaining angles in the first parallelogram are respectively equal to the remaining angles in the second.

101. THEOREM [Pm.4].—**The diagonals of a parallelogram bisect each other.**

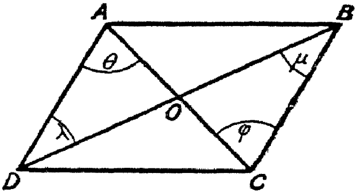


FIG. 165.

Given that $ABCD$ is a $\|^{m}$ (Fig. 165),
Required to prove that $AO = CO$, $DO = BO$.

Proof.—In $\|^{m}$ $ABCD$, $AD = BC$. Pm.1.

$\because AD \parallel BC$, $\therefore$ (using AC as trv.) $\angle\theta = \angle\phi$, P.1*c*.

and (using BD as trv.) $\angle\lambda = \angle\mu$. P.1*c*.

In $\triangle$s AOD, COB, $\because \left\{\begin{array}{l} AD = CB \\ \angle\theta = \angle\phi \\ \angle\lambda = \angle\mu \end{array}\right\}$

therefore $AO = CO$, and $DO = BO$. C.2.

COROLLARY.—**The diagonals of a rhombus, square or rectangle, bisect each other.**

102. THEOREM [Pm.5].—**The diagonals of a rhombus or square bisect its angles and are perpendicular to each other.**

Given that $ABCD$ is a rhombus or square (Fig. 166),
Required to prove that $\angle$s A, B, C, D, are bisected by the diagonals, and that $AC \perp BD$.

Proof.—

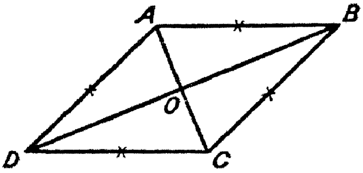


FIG. 166.

In △s ADC, ABC,

$$\because \begin{cases} AD = AB \\ CD = CB \\ AC = AC \end{cases}$$

$\therefore \angle CAD = \angle CAB$, and $\angle ACD = \angle ACB$; C.3.

that is $\angle$s A and C are bisected by diag. AC.

Similarly it can be proved that $\triangle ABD \equiv \triangle CBD$, and **hence** that $\angle$s B and D are bisected by diag. BD.

$$\text{In } \triangle\text{s } AOD, AOB, \because \begin{cases} AD = AB \\ AO = AO \\ \angle OAD = \angle OAB \end{cases}$$

$\therefore \angle AOD = \angle AOB$. C.1.

But these are adjacent angles; **hence** $AC \perp DB$.

103. THEOREM [Pm.6].—**The diagonals of a rectangle or square are equal.**

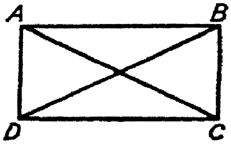


FIG. 167.

Given that $ABCD$ is a rectangle or square (Fig. 167),

Required to prove that $AC = BD$.

Proof.— $\because ABCD$ is a $\|^{m}$ Pm.1*c*.

$\therefore AD = BC$. Pm.1.

$\because ABCD$ is a rect., $\therefore \angle ADC = \angle BCD$.

$$\text{In } \triangle\text{s } ADC, BCD, \because \begin{cases} DA = CB \\ DC = CD \\ \angle ADC = \angle BCD \end{cases}$$

therefore $DB = CA$. C.1.

EXERCISES XXXVI (*Riders*).

1. In any parallelogram the diagonal which joins the angular points of the obtuse angles is shorter than the other diagonal.

2. If the diagonals of a quadrilateral bisect each other the quadrilateral is a parallelogram.

3. If the diagonals of a quadrilateral bisect each other and are equal the quadrilateral is a rectangle.

4. If the diagonals of a quadrilateral bisect each other at right angles the quadrilateral is a rhombus.

104. THEOREM [Pm.7].—**If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and have also the angles included between those sides equal, then the two parallelograms shall be congruent.**

In Fig. 168, $ABCD$, $HKLM$ are two parallelograms:—

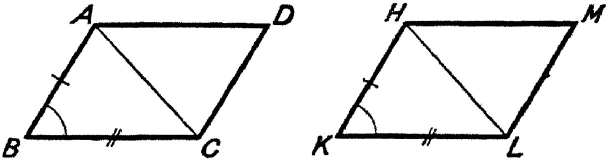


FIG. 168.

Given that $AB = HK$, $BC = KL$, $\angle B = \angle K$,

Required to prove that $\|^{\text{m}}\ BD \equiv \|^{\text{m}}\ KM$.

Cons.—Join AC, HL.

Proof.—In $\triangle$s ABC, HKL,

$$\because \left\{ \begin{array}{l} AB = HK \\ BC = LK \\ \angle B = \angle K \end{array} \right\}$$

$\therefore \triangle ABC \equiv \triangle HKL$. C.1.

In $\|^{\text{ms}}\ BD$ and KM,

$\triangle CDA \equiv \triangle ABC$, and $\triangle LMH \equiv \triangle HKL$. Pm.1.

But $\triangle ABC \equiv \triangle HKL$;

hence $\triangle CDA \equiv \triangle LMH$.

Since $\triangle$s ABC, CDA are respectively congruent to $\triangle$s HKL, LMH,

and since the relative position of the first pair of $\triangle$s in $\|^{\text{m}}\ BD$ is the same as the relative position of the second pair in $\|^{\text{m}}\ KM$,

therefore $\|^{\text{m}}\ BD \equiv \|^{\text{m}}\ KM$.

COROLLARY 1.—**If two rectangles have two adjacent sides of the one respectively equal to two adjacent sides of the other, then the rectangles are congruent.**

COROLLARY 2.—**If two squares are constructed on equal sides they are congruent.**

105. *Prel. Ex.*—In Fig. 169, $HK=KL$, $HX \parallel KY$, and $KX \parallel LY$. Prove that $HX=KY$.

THEOREM [P. 5].—**If there are three or more parallel straight lines and if the intercepts made by them on any one transversal are equal, then the intercepts made by them on any other transversal are equal.**

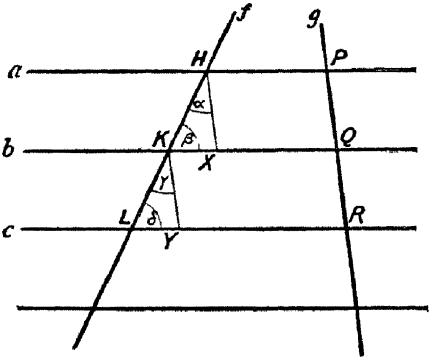


FIG. 169.

Given that $a, b, c, \ldots$ are parallel lines (Fig. 169), and that the intercepts $HK, KL, \ldots$ which they make on the transversal f are all equal;

Required to prove that the intercepts $PQ, QR, \ldots$ which they make on any other transversal g are all equal.

Cons.† — Through H and K let HX and KY be drawn each parallel to g.

Proof.—$\because HX \parallel g$ and $KY \parallel g$, $\therefore HX \parallel KY$; P.3.
hence (using f as trv.) $\angle\alpha = \angle\gamma$. P.2*c*.
$\because KX \parallel LY$, $\therefore$ (using f as trv.) $\angle\beta = \angle\delta$. P.2*c*.

$$\text{In } \triangle\text{s } HKX, KLY, \because \left\{\begin{array}{l} HK=KL \\ \angle\alpha = \angle\gamma \\ \angle\beta = \angle\delta \end{array}\right\} \therefore HX=KY. \qquad \text{C.2.}$$

$\because HP \parallel XQ$, and $HX \parallel PQ$, $\therefore HPQX$ is a $\|^{\text{m}}$;
hence $HX=PQ$. Pm.1.
Similarly it can be proved that $KY=QR$.
But $HX=KY$; **hence** $PQ=QR$.

Similarly it can be proved that PQ is equal to any other of the intercepts made by the parallel lines on the transversal g.

106. *Prel. Ex.*- In Fig. 170 AB and GE bisect each other at F. (*a*) Prove that AE, GB are equal and parallel. Hence (*b*) Given that E bisects AC, prove that FE is parallel to BC; and conversely (*c*) given that FE is parallel to BC, prove that E bisects AC.

---

† *Hypothetical construction.* See footnote on p. 105.

THEOREM [T.7].—(i) **The line which joins the middle points of the sides of a triangle is parallel to, and half the length of, the base; and (ii) the line drawn parallel to the base, from the middle point of one side, bisects the other side and is half the length of the base.**

In $\triangle ABC$ (Fig. 170), F bisects the side AB.

(i) *Given* that E bisects AC, *Required to prove* that $FE \parallel BC$, and $FE = \frac{1}{2}BC$.

(ii) *Given* that $FE \parallel BC$, *Required to prove* that E bisects AC, and $FE = \frac{1}{2}BC$.

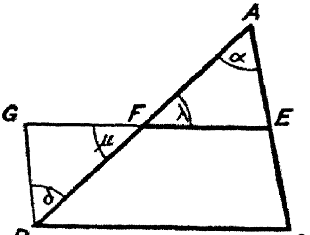


FIG. 170.

Cons.—Produce EF to G, making $FG = FE$. Join BG.

Proof.†—$\because AB$ and EG are st. lines, $\therefore \angle\lambda = \angle\mu$. A.2.

$$\text{In } \triangle\text{s } AFE,\ BFG,\ \because \left\{\begin{array}{l} AF = BF \\ EF = GF \\ \angle\lambda = \angle\mu \end{array}\right\}$$

$\therefore \angle\alpha = \angle\delta$ and $AE = BG$. C.1.

$\because \angle\alpha = \angle\delta$, $\therefore$ (using AB as trv.) $AC \parallel BG$. P.1.

(i) $\because BG = AE$ (*Proved*), and $AE = CE$ (*Hyp.*), $\therefore BG = CE$.

In quad. $GBCE$, $\because BG = CE$ and $BG \parallel CE$,

$\therefore GE \parallel BC$ and $GE = BC$. Pm.2.

Hence $FE \parallel BC$, and $FE = \frac{1}{2}BC$.

(ii) $\because BG \parallel CE$ (*Proved*), and $GE \parallel BC$ (*Hyp.*);

$\therefore BGEC$ is $\parallel^{m}$, whence $BG = CE$, and $GE = BC$. Pm.1.

$\because BG = CE$, and $BG = AE$, **therefore** $AE = EC$,

$\because GE = BC$, and $FE = \frac{1}{2}GE$, **therefore** $FE = \frac{1}{2}BC$.

NOTE.—The first portion of Part (ii) of this theorem can be proved at once from Theorem P.5. For if AX be drawn $\parallel BC$ (Fig. 170), then AX, FE, BC are parallels; also $AF = FB$. Hence $AE = EC$, by P.5.

DEFINITION.—**In any triangle any straight line which joins the middle point of a side to the opposite vertex is called a median.**

Thus in Fig. 170 the lines BE, CF would be medians of the triangle ABC.

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† The first portion of the *Proof* applies both to Part (i) and Part (ii).

EXERCISES XXXVII.

RIDERS.

(*In Questions* 1–4 *the references indicate all the Propositions required in the proof.*)

1. In the quadrilateral *ABCD, AB* is parallel to *CD.* Prove that the line drawn through the middle point of *AD* parallel to *AB* bisects the opposite side and both diagonals. P.5.

2. In the quadrilateral *ABCD, AB* is parallel to *CD,* and the sides *AD, BC* are bisected at the points *H, K.* Prove that *HK* is parallel to *AB* or *CD,* and equal to $\frac{1}{2}(AB + CD)$. Playfair's Axiom ; P.5.; T.7.

3. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles. P.2*c* ; T.1.

4. Straight lines bisecting two opposite angles of a parallelogram are either coincident or parallel. P.1*c*.; Pm.1 ; P.2.

5. If the straight line, joining two opposite angles of a parallelogram, bisects the angles, the parallelogram is a rhombus. P.1*c*; T.2*c*.

6. *ABCD* is a parallelogram and *BD* is a diagonal. Prove that the perpendiculars from *A* and *C* on *BD* are equal. C.2.

7. Prove that if two lines are parallel, the distance from either line of any point on the other line is constant, *i. e.* **is the same for all points on these lines.**† P.2. ; P.2*c*.

8. In a rhombus the distance between one pair of parallel sides is equal to the distance between the other pair of parallel sides. C.2.

9. If *D, E, F* are the middle points of the sides of an equilateral triangle prove that the triangle *DEF* is equilateral.

10. If *D, E, F* are the middle points of the sides *BC, CA, AB* of a triangle *ABC,* the triangles *DEF, AEF, BFD, CDE* are all congruent.

11. *ABCD* is a parallelogram ; *BK, DL* are two parallel lines meeting the diagonal *AC* in *K* and *L ;* prove that $AK = CL$, $\triangle ADL \equiv \triangle CBK$.

12. In the quadrilateral *ABCD, AB* is parallel to *DC;* also *AD* is trisected at *E, F,* and *BC* at *H, K.* Prove that *EH* is parallel to *FK.*

13. If *D* be the middle point of the base *BC* of an isosceles triangle *ABC,* and *E* and *F* be the middle points of the sides, prove that *AD* is perpendicular to *EF,* bisects it, and is bisected by it.

14. If the angle between two adjacent sides of a parallelogram be increased, while their lengths do not alter, the diagonal through the point of intersection will decrease.

15*. *ABCD* is a parallelogram, and *E, F, G, H* are the middle points of the sides *AB, BC, CD, DA* respectively. Prove that *EG* and *FH* are parallel to *BC* and *CD* respectively, and that *EFGH* is a parallelogram. (Prove and use the converse of Pm.4.)

---

† This distance is called the **distance between the parallel lines.**

16*. In the preceding question if $ABCD$ is a rhombus then $EFGH$ is a rectangle and *vice versâ*; and if $ABCD$ is a square, then $EFGH$ is a square. (Prove and use the converses of Pm.5 and Pm 6.)

17*. In any parallelogram the distance between the longer sides is less than the distance between the shorter sides. (Cut off a rhombus whose sides are equal to the shorter side of the parallelogram.)

18*. In a quadrilateral $ABCD$, $\angle A=\angle B$, $\angle C=\angle D$. Prove that AB is parallel to CD, and AD equal to BC.

19*. In a quadrilateral two opposite sides are equal, and the other two opposite sides are parallel. Prove that opposite angles are either equal or supplementary.

PROBLEMS IN PRACTICAL GEOMETRY.

20. Construct a parallelogram in which two adjacent sides measure 3·6 and 4·8 cms. respectively, and one diagonal measures 6 cms. Measure the other diagonal.

21. Construct a triangle ABC, given $AB=1\cdot8''$, $BC=3''$, $\angle B=60^\circ$. Complete the parallelogram $ABCD$ having AB, BC for adjacent sides. Measure the diagonal BD.

22. Construct a parallelogram in which one side measures 5 cms. and the two diagonals measure 6 and 10 cms. respectively. Measure the other sides.

23. Construct a parallelogram in which the diagonals measure 6 cms. and 10 cms. respectively and are inclined to one another at an angle of 60°. Measure the distance between the longer sides.

24. Construct a parallelogram in which two adjacent sides measure 1·5″ and 2·5″, and the distance between the pair of longer sides is 1·2″. Measure the distance between the other pair of sides.

25. Construct a parallelogram in which two adjacent sides measure 3 and 5 cms., and the distance between the pair of longer sides is as great as possible. Measure the two diagonals.

26. Construct a rectangle having one side of length 1·5″ and a diagonal of length 2·4″. Measure the angle between the diagonal and the longer side.

27. Construct a rhombus having each side and one diagonal 1″ in length. Measure the other diagonal.

28. Construct a rhombus whose diagonals measure 3 and 5 cms. Measure the distance between either pair of opposite sides, and also the acute angle of the rhombus.

29. Construct a rhombus in which the longer diagonal measures 7 cms. and the acute angle measures 50°. Measure the other diagonal.

30. Construct a rhombus each of whose sides measures 6 cms., and in which opposite sides are at a distance of 4 cms. from each other. Measure the obtuse angle,

31. Construct a square whose diagonals measure 4 cms. and measure the sides.

32. State the converse of Theorem P.5. Draw a figure showing clearly that the converse does not hold.

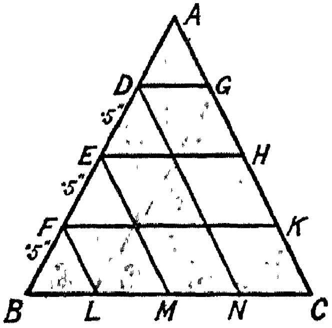

FIG. 171.

33. Copy Fig. 171 accurately *to the given dimensions.* *ABC* is an equilateral triangle; *DG, EH, FK* are all ∥ *BC*; *DN, EM, FL* are all ∥ *AC*. Join *GL, HM, KN*. The figure should now be accurately divided into 16 equilateral triangles of sides ·5".

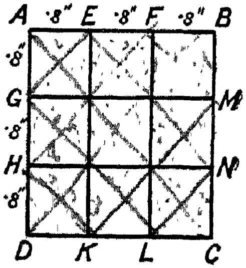

FIG. 172.

34. Copy Fig. 172 accurately *to the given dimensions.* *AB* is ⊥ *AD*; *EK, FL, BC* are all ∥ *AD*; *GM, HN, DC* are all ∥ *AB*. The figure now consists of 9 equal squares. Join *EG, FH, BD, MK, NL, HK, GL, AC, EN, FM*. The figure should now be accurately divided into 36 equal right-angled isosceles triangles.

CHAPTER V.

PROBLEMS.

107. THE Propositions in this chapter are all Problems, that is to say their object is to establish the validity of certain geometrical constructions. Most of these constructions have already been given in the Introductory Course; but for the student's convenience these are usually repeated here in connection with the formal proof which justifies them.

Revise § 10 of the Introductory Course (pp. 10, 11), and commit the definitions to memory.

Note that the statement that all radii of a circle are equal is implied in the definition of the circle.

PROBLEM.†—**To bisect a given angle.**

Required to bisect $\angle ABC$ (Fig. 173).

Cons.—Using the compasses mark off from BA, BC, any equal lengths BD, BE.

With centres D, E, and any convenient equal radii draw two arcs cutting at F. BF will bisect $\angle ABC$.

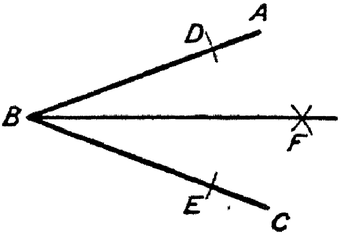

FIG. 173.

Proof.—Join DF, EF.‡

In $\triangle$s BDF, BEF, $\because \left\{\begin{array}{l} BD=BE, \\ DF=EF, \\ BF=BF, \end{array}\right\}$ **therefore** $\angle DBF = \angle EBF$. C.3.

---

† As the use of Hypothetical Constructions is now permitted, the Problems do not form an *essential* part of the logical course; accordingly they have not been included in the scheme of marginal references, but are referred to by means of the section number—*e. g.* this Problem is indicated by the reference "§ 107." For this reason too the proofs of Problems are printed in small type. The authors, however, advise that students should learn the proofs of Problems as well as of Theorems, and in writing out these proofs should insert references to Theorems *only*.

‡ These lines are required only in the Proof.

108. **On accuracy in drawing.**—In many geometrical constructions certain points are determined by the intersection of two lines, straight or curved, as for instance in the case of the point F (Fig. 173).

In such cases there are two possible sources of error:—

(i) One or both of the intersecting lines may be slightly out of position (*e.g.* in Fig. 173 we may not have used the exact points D and E for centres);

(ii) the two lines may intersect at such a small angle that it is difficult to determine the exact point where they cut.

The possible error due to (i) is least when the intersecting lines meet at right angles†; it is about twice as great if they intersect at 45° and is seriously increased if they intersect at a smaller angle. If the lines are thin, the error due to (ii) is of no importance unless the acute angle at the intersection is about 20° or less. Accordingly the possible effect of errors due to both causes will be reduced to a minimum if it can be arranged that the *intersecting lines should cut roughly at right angles.* Thus in Fig. 173 we can get a very good intersection at F by taking the radius DF rather less than DE.

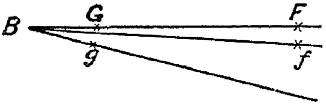

FIG. 174.

In many constructions however it is impossible to vary the angle at which the two lines intersect, and in others the right-angled intersection could only be obtained by means of an awkward or unwieldy figure. The following is therefore the best working rule:—

If a point is to be obtained by the intersection of two lines, it is advisable that these lines should not form an angle of less than 45°.

Fig. 173 will also serve to illustrate another important point:—The arcs drawn with centres D and E would also intersect at some point G to the left of the line DE; and *theoretically* the angle B would be bisected as accurately by joining BG as by joining BF. In *practice*, however, the point G is much too near the point B to give a good result. For a slight error in the position of G would have much more effect on the direction of the line BG, than an equal error in the position of F would have on the direction of the line BF. See Fig. 174 where BGF is the true position of the bisector, and f, g are at equal ‡ distances from F, G respectively. In general:—

In practical geometry a line cannot be accurately determined by means of two points upon it which are near together.

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† The reason for this is rather too difficult for the student at this stage.

‡ To the eye Ff appears smaller than Gg—an interesting optical illusion. Each length is ·2 cm.

109. PROBLEM.—**At a given point in a given straight line to construct an angle equal to a given angle.**

Required to construct at E in the given line ED an angle equal to the given angle ABC (Fig. 175).

Cons.—With centres B, E and any convenient equal radii draw two arcs FG, HK, intersecting BA, BC, ED at F, G, H respectively.

With centre H and radius FG† describe an arc cutting the arc HK at K.

Join EK. Then DEK shall be the required angle.

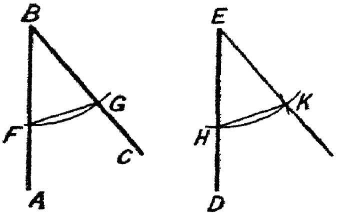

FIG. 175.

Proof.—Join FG, HK.

In △s EHK, BFG, $\because \left\{\begin{array}{l} EH=BF \\ EK=BG \\ HK=FG \end{array}\right\}$ therefore $\angle E = \angle B$. C.3.

110. At this stage the class should revise §§ 12, 23, 24 of the *Introductory Course*, which are of very great importance but require no proof. In practising the problems in these sections the class should dispense with a protractor, using only angles given on a diagram.

To arrange this work for a class the teacher can first instruct the pupils to construct the following angles by means of the protractors :—$P=50°$, $Q=65°$, $S=70°$, $T=75°$, $X=80°$, $Z=90°$. After the protractors are put away the following exercises can be worked :—

EXERCISES XXXVIII (Practical Geometry).

Construct triangles ABC from the following data :—

1. $AB=AC=2''$, $\angle A=\frac{1}{2}X$. Measure BC.
2. $AB=4$ cms., $\angle A=P$, $\angle B=Z$. Measure BC.
3. $AB=2''$, $\angle A=\angle B=\frac{1}{2}S$. Measure AC.
4. $AB=2$ cms., $AC=2{\cdot}5$ cms., $\angle B=Z$. Measure BC.
5. $AB=1{\cdot}5''$, $\angle A=\frac{1}{2}Z$, $\angle B=T$. Measure BC.
6. $BC=1{\cdot}5''$, $\angle B=\frac{1}{2}X$, $\angle C=Q$. Measure AB.
7. $AB=AC=2''$, $\angle A=\frac{1}{4}X$. Measure BC.
8. $BC=1{\cdot}5''$, $AC=1''$, $\angle A=S$. Measure AB.
9. $BC=1''$, $AC=1{\cdot}25''$, $\angle A=\frac{1}{2}S$. Measure AB.
10. $AC=6{\cdot}6$ cms., $AB=4{\cdot}4$ cms., $\angle C=\frac{1}{2}X$. Measure BC.

Construct quadrilaterals $ABCD$ from the following data :—

11. $AB=BC=2''$, $CD={\cdot}6''$, $\angle B=\angle C=Z$. Measure AD.
12. $AD=2{\cdot}1''$, $AB=1{\cdot}4''$, $BC=1''$, $CD=1''$, $\angle A=P$. Measure AC.

---

† The compass legs are to be set to the points F and G. The radius will then be equal to the *straight line* FG.

111. PROBLEM.—**To draw a line through a given point parallel to a given straight line.**

Required to draw through the given point P a line parallel to the given line AB (Fig. 176).

FIRST METHOD: *Cons.*—Take any convenient point C in the line AB.

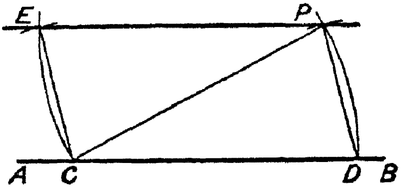

FIG. 176.

With centre C and radius CP† describe the arc PD, cutting AB in D.

With centre P and radius PC describe the arc CE. With centre C and radius DP describe an arc cutting arc CE at E. Join PE.

Then shall EP be parallel to AB.

Proof.—Join CP, PD, CE.
$\because EP = CP$, and $CP = CD$ (*Cons.*), $\therefore EP = CD$.
In quad. $EPDC$, $\because EP = CD$ and $EC = PD$, $\therefore EPDC$ is a $\|^{m}$. Pm.1*c*.
That is $EP \parallel CD$.

SECOND METHOD: See the method by set squares given on p. 41 (Fig. 56). Note that CM really *coincides with* AB and HD with EG.

Proof.—Since the angles DCM, LHK are equal to the same angle on the set square, $\therefore \angle DCM = \angle LHK$.
Hence (using HD as trv.) $HK \parallel AB$. P.2*c*.

NOTE.—*Theoretically* there is no reason to prefer either of these methods to the other. In *practice* the second method is usually the more convenient, especially if P lies very near AB; but the second method is not accurate if the edges EG, CD of the set squares deviate sensibly from the straight.

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† See footnote on p. 149.

112. PROBLEM.—**To draw a line perpendicular to a given straight line from a given point within it.**

Required to draw a line from $C \perp AB$ (Figs. 177 and 178).

FIRST METHOD: *Cons.*—Using the compasses mark off on the line AB on each side of the point C any convenient equal lengths CD, CE (Fig. 177).

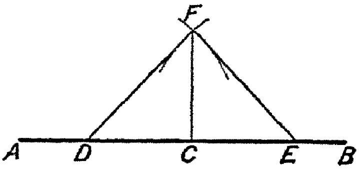

FIG. 177.

With centres D and E and any convenient equal radii draw two arcs intersecting at F. Join CF.

Then CF shall be the required perpendicular.

Proof.—Join DF, EF.

In $\triangle$s DFC, EFC, $\because \begin{cases} DF=EF \\ DC=EC \\ FC=FC \end{cases}$ $\therefore \angle FCD = \angle FCE$. C.3.

But these are adjacent angles, **hence** $CF \perp AB$.

SECOND METHOD*: *Cons.*—Take any convenient point O outside the line (Fig. 178).

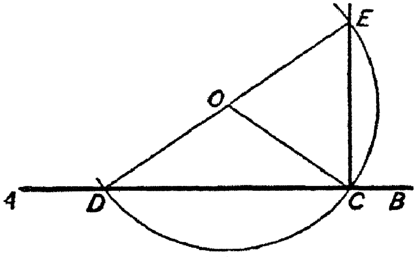

FIG. 178.

With centre O and radius OC describe the circle DCE, cutting AB a second time at D.

Join DO and produce to meet the circumference again at the point E. Join CE.

Then CE shall be the required perpendicular.

Proof.—Join OC.

In $\triangle OCE$, $\because OE=OC$, $\therefore \angle OCE = \angle OEC$. T.2.

In $\triangle OCD$, $\because OD=OC$, $\therefore \angle OCD = \angle ODC$. T.2.

Add these equals, $\therefore \angle ECD = \angle OEC + \angle ODC$.

$\because ECB$ is an exterior angle of $\triangle DCE$,

$\therefore \angle ECB = \angle OEC + \angle ODC$. T.1.

Hence $\angle ECD = \angle ECB$.

But these are adjacent angles; **hence** $EC \perp AB$.

THIRD METHOD.—By the use of the set squares, as described on p. 23. Note that KL really *coincides with* AB, and HR with MO.

Proof.—Because each of the angles QPR, KHL is equal to the same angle of the set square,

$$\therefore \angle QPR = \angle KHL.$$

Hence (using HL as trv.) $PQ \parallel HK$. P.2*c*.
Thus (using KL as trv.) $\angle PCL = \angle HKL$. P.2.

But, if the set square is accurate, $\angle HKL = \text{R}$. **Hence** $\angle PCL = \text{R}$.

NOTE 1.—Of these three methods, the first two are *theoretically* perfect. In practice the first is very accurate, as the exact position of the various points of intersection can be easily determined; DF should be rather less than DE. The second is more convenient when C lies near one extremity of AB, as it does not require the line to be produced; also the second is as accurate as the first provided that $\angle OCA$ is about 45°, and O is not too near to C; the value of this rule is easily verified by repeating the construction several times with O in different positions. The accuracy of the third method depends upon the set square.

NOTE 2.—To test the accuracy of the right angle, take a point X on AB at a distance of 4 cms. from C, and a point Y on the perpendicular at a distance of 3 cms. from C. Then XY should measure *exactly* 5 cms. See R.T.2. The fact that a triangle with sides in the ratio 3 : 4 : 5 was right-angled was known to the Egyptians at least 2000 years B.C.

113. PROBLEM.—**To bisect a given straight line.**

Required to bisect AB (Fig. 179).

Cons.—With centres A, B, and any convenient equal radii, draw two arcs intersecting at C and D.

Join CD by a line meeting AB at P. Then P shall be the middle point of AB.

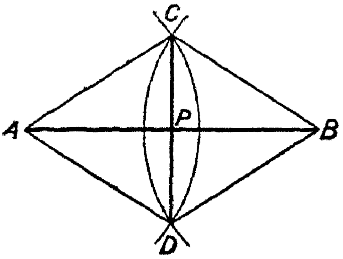

FIG. 179.

Proof.—Join AC, AD, BC, BD.

Since AC, AD, BC, BD are all equal, $ADBC$ is a rhombus and its diagonals bisect each other. Pm.5.

Hence $AP = BP$.

COROLLARY.—**CD is perpendicular to AB.** Pm.5.

Caution.—In practice only the portions of the arcs near C and D are drawn, the lines AC, AD, BC, BD are not joined and the point P is determined by laying the straight edge between C and D, not by actually drawing CD. Similar remarks apply to most of the figures in this section. It is a good exercise for the pupil to point out exactly which lines in the figure may be omitted.

114. PROBLEM.—**To draw a line perpendicular to a given straight line from a given point without the line.**

Required to draw a line from $C \perp AB$ (Figs. 180 and 181).

FIRST METHOD.—With centre C and any convenient radius draw an arc cutting AB at the points D, E (Fig. 180).

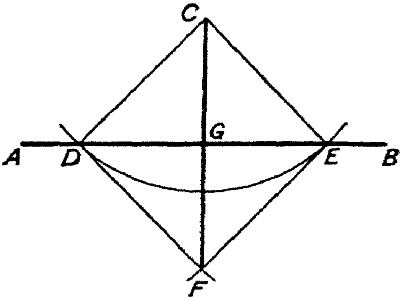

FIG. 180.

With centres D, E, and any convenient equal radii draw two arcs intersecting at F.

Join CF, cutting AB at G. Then CF will be the required perpendicular.

Proof.—Join CD, CE, DF, EF.

$$\text{In }\triangle\text{s } DCF,\ ECF,\ \because \begin{cases} CD = CE \\ DF = EF \\ CF = CF \end{cases} \therefore \angle DCF = \angle ECF. \qquad \text{C.3.}$$

$$\text{In }\triangle\text{s } DCG,\ ECG,\ \because \begin{cases} CD = CE \\ CG = CG \\ \angle DCG = \angle ECG \end{cases} \therefore \angle CGD = \angle CGE. \qquad \text{C.1.}$$

But these are adjacent angles; hence $CG \perp AB$.

SECOND METHOD: *Cons.*—Join C to any convenient point D in AB (Fig. 181).

Bisect CD at O. § 113.

With centre O and radius OC (or OD) describe a semi-circle meeting AB at G. Join CG.

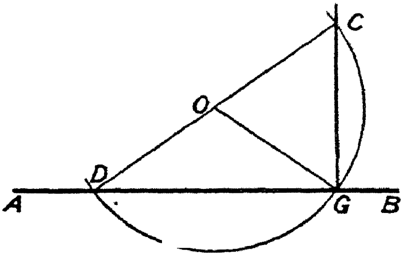

FIG. 181.

Then CG is the required perpendicular.

The proof is similar to that of the second method given in § 112.

THIRD METHOD.—See the method by set squares given on p. 25.

The proof is similar to that of the third method given in § 112.

NOTE.—Of these three methods the first two are theoretically perfect; there is very little to choose between them in point of accuracy. In both methods $\angle CDB$ should not be greater than 45°; the value of this rule is easily verified by repeating the constructions with D in various positions. The first method is more convenient when G will fall near the middle of the line, and the second method when G will fall near either extremity. The accuracy of the third method depends upon the set square.

115. PROBLEM.—**To divide a given straight line into five equal parts.**

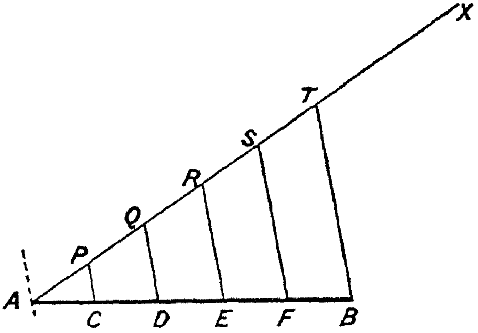

FIG. 182.

Required to divide AB into 5 equal parts (Fig. 182).

Cons.—Draw any line AX at a convenient angle with AB. Along AX mark off with the compasses 5 equal lengths AP, PQ, QR, RS, ST of any convenient size. Join BT.

Through $P, Q, R, S,$ draw lines $PC, QD, RE, SF \parallel BT$.

These lines will divide AB into 5 equal parts.

Proof.—Suppose a line drawn through A parallel to BT. We have then a series of parallel lines and two transversals AX, AB.

But the 5 intercepts $AP, PQ, \ldots$ which the parallel lines make on the transversal AX are all equal.

Hence the 5 intercepts $AC, CD, \ldots$ which the parallel lines make on the transversal AB are all equal;

that is AB is divided into 5 equal parts.

NOTE 1.—A straight line can be divided into any number of equal parts by a similar construction.

NOTE 2.—No simple rules can be given for drawing a good figure. The following hints will be found useful:—$\angle A$ may vary roughly from 30° to 70° the latter being useful when AB is *short.* Try to choose $\angle A$ and length AP so that $\angle B$ will lie roughly between 60° and 120°. AP should not be less than about $\frac{1}{2}$ inch, as smaller lengths are not easily marked off with the compasses ; if $\frac{1}{2}$ inch is obviously too large, mark off shorter equal lengths from the scale of millimetres, and in using this scale be careful to avoid the parallax error. (See § 6, Introductory Course.)

116. PROBLEM.—**To construct a square given one side.**

Required to construct a square having AB for one side (Fig. 183).

Cons.—From A draw $AC \perp AB$. From AC cut off AD equal to AB. With centres D and B and radii equal to AB, describe two arcs intersecting at the point E.

Join DE, BE.

Then $ABED$ shall be the required square.

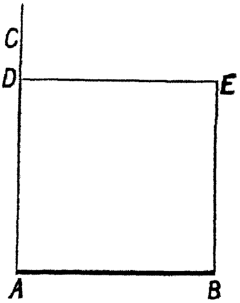

FIG. 183.

Proof.—Each of the other three sides is equal to AB.

Also $\angle A = \mathrm{R}$.

Hence $ABCD$ is a square. *Def.*

EXERCISES XXXIX.

RIDERS.

In Questions 1–4, 7–10 *the student must give a construction for th problem which does not involve the use of protractor or set squares, anc must prove that this construction is correct.*

In Questions 1–6 *the references indicate* **all** *the Propositions required.*

1. Draw a straight line through a given point P which shall mak with a given straight line AB an angle equal to a given angle C. § 109, § 111, P.3c

2. Through a given point draw a straight line so that the par intercepted between two given parallel straight lines may be equal to given straight line. § 111, Pm.1

3. Draw a straight line parallel to the base of a triangle so that the portion intercepted between the sides may be equal to a given straight line. § 111, Pm. 1.

4. Draw a straight line through a given point making equal angles with two given intersecting straight lines. Can more than one such line be drawn? § 111, P. 2c, T. 2.

5. Prove the following construction for trisecting a right angle ABC, *i.e.* dividing it into three equal parts:—

With centre B and any convenient radius draw an arc intersecting BA, BC at D, E respectively; with centres D, E and the same radius as before draw arcs meeting the arc DE at F, G; join BF, BG. (*Note that DBF is an equilateral triangle.*) T. 1, T. 2.

6. A point is to be determined by the intersection of two straight lines. One of them is drawn accurately, and the other is drawn parallel to its correct position at a distance of ·1 inch. Show that the resulting error in the position of the point is least when the two lines are at right angles. P. 2c, T. 5.

7. Construct a quadrilateral figure equal in all respects to a given quadrilateral figure.

8. Construct a parallelogram given the lengths of the diagonals and the angle between them.

9. Construct a rhombus given the lengths of the diagonals.

10. Construct a rhombus given the length of one side and the perpendicular distance between this side and the opposite side.

11. Prove the following construction for drawing a line from B perpendicular to the line AB:—On AB describe an equilateral triangle ABC; produce AC to D, making CD = AC; join BD.

12. Prove the following construction for trisecting a line AB:—On AB as base describe an equilateral triangle ABC; bisect the angles A, B by two lines intersecting at D; through D draw lines parallel to CA, CB, meeting AB at E, F.

13.* Prove the following construction for bisecting an angle ABC:—Take any two points H, L, on BA; on BC mark off lengths BK, BM equal to BH, BL respectively; join HM, KL, intersecting at N; join BN.

14.* Prove the following construction for dividing a line AB into five equal parts:—Through A draw any line AX making a convenient angle with AB; through B draw BY parallel to and in the opposite sense to AX; from A mark off four equal parts AC, CD, DE, EF, along AX; from B mark off four parts BG, GH, HK, KL along BY, each part equal to AC. Join CL, DK, EH, FG, meeting AB in P, Q, R, S respectively. Pm. 2, P. 5.

PROBLEMS IN PRACTICAL GEOMETRY AND SCALE DRAWING.

(*The following Questions require very exact drawing; protractor and set squares are not to be used.*†)

15. Take a line AB of length 1 cm. Find two points C, D each of which is 5 cms. from A, and 4·1 cms. from B. Measure CD.

16. Construct a triangle PQR, making PQ, QR, RP equal to ·3″, ·5″, ·7″ respectively. Produce QR to S, making $RS=QR$. Draw ST perpendicular to PQ. Measure QT.

17. Draw a line AB of length 3″. From BA cut off $BC=$ ·3″. Draw BX, CY, perpendicular to AB, and on the same side of it, making $BX=$ 2·5″, $CY=$ 2·2″. Join XY, and produce it to meet AB at Z. Measure AZ.

18. Draw a line PQ of length 2″. From P, Q draw PX, QY perpendicular to PQ and on opposite sides of it. Divide PQ into 7 equal parts, and let R be the dividing point nearest to P. From PX mark off $PS=$ ·3″. Join SR and produce it to meet QY at T. Measure QT.

19. Draw AB of length ·5″; at A and B erect perpendiculars AX, BY on the same side of AB. From BY mark off $BC=$ ·2″; join AC and divide it into 5 equal parts. Join BD, where D is the dividing point on AC nearest to C. Produce BD to meet AX at E. Measure AE.

20. Construct a seven-sided figure $OABCDEF$ from the following data:—$OA=AB=BC=CD=DE=EF=$ 1″; $\angle OAB=\angle OBC=\angle OCD=\angle ODE=\angle OEF=$ R. Measure OD, OE, OF.

21. There are three church spires A, B, C. A is 1 mile from B; C is ·7 mile from A and $\frac{1}{3}$ mile from B. How far does C lie from the straight road AB?

22. P, Q are two consecutive mile-stones on a straight road running due north. A tower T is visible from both points; at P it bears "north by east," (*i.e.* $11\frac{1}{4}°$ east of north), at Q it bears "north-north-east" (*i.e.* $22\frac{1}{2}°$ east of north). Find the distance of the tower from the road PQ.

23. A ship is exactly 10 miles due north of a rock. How far must it sail due east in order to be exactly $10\frac{1}{3}$ miles from the rock?

24. A, B, C, D, E, F, G, H are telegraph-posts occurring at intervals of $\frac{1}{7}$ furlong along a road bearing 30° N. of E. X is a church-spire $\frac{1}{3}$ furlong due north of A; Y is another spire due south of H. If XBY is a straight line, determine the distance HY.

---

† In Questions 15–20 a good system of marking would be to give 4, 3, 2, 1 marks for answers in which the error is not greater than ·02″, ·04″, ·06″, ·08″ respectively. In Questions 21–24 give 4, 3, 2, 1 marks for answers in which the error is not greater than 1, 2, 3, 4 per cent. respectively.

CHAPTER VI.

LOCI.

117. In working through the following exercises, remember that the distance of a given point from a given line is measured by the length of the *perpendicular* drawn from the given point to the given line *or to the given line produced.*

EXERCISES XL (Oral†).

1. Take any point *A*. Find ten different points each of which is exactly 1·2″ from *A*.

(*a*) On what line or curve do these points all lie?

(*b*) Is every point on this curve at a distance of 1·2″ from *A*?

(*c*) Is every point (on the paper) which is 1·2″ from *A* on this curve?

2. Take any line *AB*. Find ten points each of which is ·7″ from *AB* (some on each side of *AB*).

(*a*) On what pair of lines do these points all lie?

(*b*) Is every point on either of these two lines at a distance of ·7″ from *AB*?

(*c*) Is every point (on the paper) which is ·7″ from *AB* on one of these two lines?

3. Draw an angle *HKL* about 40°. Find (by trial) a point within this angle, which is equidistant from *HK* and *KL*—*i. e.* which is the same distance from *HK* as from *KL*. Find (by trial) four other points each of which is equidistant from *HK* and *KL*.

(*a*) On what line do these points all lie?

(*b*) Is every point on this line equidistant from *HK* and *KL*?

(*c*) Is every point within the angle which is equidistant from *HK* and *KL* on this line?

4. Draw a line *AB* about 1″ long. Find a point on *AB* which is equidistant from the two points *A* and *B*—*i. e.* which is the same distance from *A* as from *B*. Find (by trial or otherwise) a point on one side of *AB* which is equidistant from *A* and *B*. Find five other points each of which is equidistant from *A* and *B*— some on one side of *AB* and some on the other.

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† These questions are best worked orally on the blackboard, the dimensions being suitably altered, *e.g.* by reading "feet" or "decimetres" for "inches."

(*a*) On what line do these points all lie?

(*b*) Is every point on this line equidistant from *A* and *B*?

(*c*) Is every point (on the paper) which is equidistant from *A* and *B* on this line?

5. Draw two parallel lines *PQ*, *RS* at a distance of about one inch. Find (by trial or otherwise) five different points each of which is equidistant from the lines *PQ* and *RS*.

(*a*) On what line do these points all lie?

(*b*) Is every point on this line equidistant from the lines *PQ* and *RS*?

(*c*) Is every point (on the paper) which is equidistant from *PQ* and *RS* on this line?

118. **Loci.**—In each problem in the preceding exercises a set of points was drawn all of which satisfied some given condition (*e.g.* in Question 4, they were all equidistant from *A* and *B*). In each case it was found (i) that these points all lay on some definite line or curve (on a *pair* of lines in one case), (ii) that all points on this line or curve (or pair of lines) satisfied this given condition, and (iii) that *no other points* on the paper satisfied this given condition.

Lines or curves which are determined in this manner are called **loci**, and this method of drawing them is described as **plotting**.

DEFINITION.—**The *locus*† of a point which satisfies a given condition is the line (straight or curved) or set of lines, upon which the point *must* lie if it satisfies this given condition, and upon which if the point *does* lie it must satisfy the given condition.‡**

119. **Example.**—*A and B are two fixed points at a distance* 1·4″. *Plot the locus of a point which moves so that the sum of its distances from A and B is always* 1·8″.

Evidently there is one point on the locus such that the distances from A and B are equal, each being ·9″. To find this point (P) draw arcs with centres A and B and radii ·9″. These arcs will intersect in two points P and Q each of which is on the locus (Fig. 184).

---

† A Latin word meaning *position* (which mathematicians usually mispronounce as *lōkus* in the singular and *lōsī* in the plural).

‡ This definition can also be expressed as follows.—

The *locus* of a point which satisfies a given condition is the line (straight or curved) or set of lines upon which it is *necessary and sufficient* that the point should lie in order that it may satisfy the given condition.

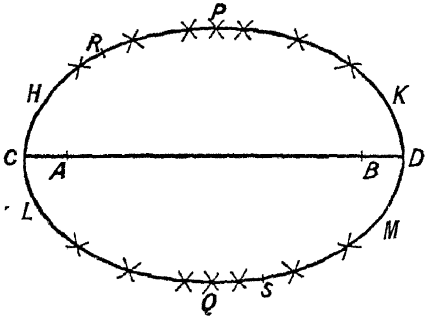

FIG. 184.

For other points on the locus we may take the series of corresponding values :

Distance from A....	·4″	1·4″	·6″	1·2″	·8″	1·0″
Distance from B....	1·4″	·4″	1·2″	·6″	1·0″	·8″

Thus arcs described with centres A and B and radii 1·4″ and ·4″ respectively will intersect at two points one above and one below AB, each of which is on the locus (Fig. 184).

Similarly each pair of values gives two points on the locus; and all these points are shown on Fig. 184.

Through these points *a curve*, as regular as possible, *must be drawn by hand.* A postcard held at right angles to the paper and bent so as to pass through the points will be found a help.

The result is, apparently, two curves, HPK and LQM, one above and the other below AB. In such cases always examine carefully whether there is any part of he locus which joins these two together.

It is clear that if we produce *AB* to *D* so that *BD* = ·2″, then *AD* = 1·6″; so that *AD* + *BD* = 1·8″, whence *D* is a point on the locus. Similarly if we produce *BA* to *C* so that *AC* = ·2″, then *AC* + *BC* = 1·8″, whence *C* is a point on the locus. The points *C* and *D* suggest that the two curves previously obtained should be joined by curves passing through *C* and *D*, the whole making a closed oval curve of the class called ellipses.†

NOTE 1.—Test the curve thus drawn by taking any other points *R* and *S* on it; it will be found, if the drawing is good, that *AR* + *BR* = 1·8″ and *AS* + *BS* = 1·8″.

NOTE 2.—Find only points enough to enable you to determine the general shape of the *complete* curve and to draw it with fair accuracy.

NOTE 3.—It is usually advisable, when plotting any locus, to commence by forming a table of corresponding values, similar to the table given above.

120. **Various Types of Loci.**—Any law which a point can satisfy in different positions may be taken as a locus-condition. By taking different conditions an endless variety of loci can be obtained. The following hints on a few common types of loci are worthy of notice:—

(1) In many cases the locus is a straight line; *as a rule* this line may be produced to any length in either direction.

(2) Sometimes the locus is a closed curve such as a circle or an oval; in such cases it will be found that neither of the measurements upon which the locus depends can be greater than a certain limit.

For instance, in the example in § 119, it is *obvious* that the distance of the locus point from either *A* or *B* must not exceed 1·8″, and *a few trials* will soon show that its distance from either *A* or *B* cannot exceed 1·6

(3) Sometimes the locus is a curve (or curves) extending indefinitely in certain directions (just as any straight line extends indefinitely in two directions).

In such cases find a sufficient number of points on the locus (i) to determine its general shape, and (ii) to draw *accurately* that portion of the locus in which the distances involved are not inconveniently large.

---

† Do not suppose that any closed oval is an ellipse.

121. In many loci the locus-condition involves only the distances of a point measured from two fixed perpendicular lines. In such cases the use of **squared paper** (tenth-inch squares) is a great advantage.

Two of the lines ruled on the paper should be taken as the given fixed lines. It is then very easy to find a point at given distances from these lines (*e. g.* ·7″ from one line and 1·2″ from the other).

EXERCISES XLI † (*Practical Geometry*).

1. Draw a right angle XOY. Plot the locus of a point within the angle the sum of whose distances from the arms of the angle is 1·6″.

Draw four lines from O to the locus inclined to OX at angles (i) 20°, (ii) 40°, (iii) 50°, (iv) 70°. Measure each of these lines.

2. Plot the locus of a point between the arms of a right angle XOY which moves so that its distance from the arm OX is always double of its distance from the arm OY.

If the points P, Q lie on OX at distances 1″, 2″ respectively from O, find the shortest distance of each point from the locus.

3. Plot the locus of a point which moves between the arms of a right angle XOY in such a way that its distance from OX is always greater than its distance from OY by ·7″.

Draw a line which cuts off lengths of 1″ from OX and OY, and a second which cuts off lengths of 2″ from OX and OY; if this line meets the locus at L and M, measure the distances of L and M from OX and OY.

4. Plot the locus of a point which moves between the arms of a right angle XOY in such a way that, if its distances from the two arms are measured in *tenth-inches*, the product of these distances is always 24. [Thus the point which is 3 tenth-inches from one arm and 8 tenth-inches from the other will be on the locus.]

Along OX measure off OA, OC equal to 12 and 15 tenth-inches respectively; along OY measure off OB, OD equal to 12 and 15 tenth-inches respectively. Join AB, CD, and measure the chord which the locus intercepts on each of these lines.

---

† In this set of exercises the second part of each question is merely intended as a test for the accuracy of the locus-drawing. Where the locus is a straight line the usual degree of accuracy may be demanded. But in other cases an error of ·04″ may be allowed, and an error of ·08″ may receive half marks.

5. Take two points A, B at a distance of 1″. Plot the locus of a point the sum of whose distances from A and B is 2″.

Draw three lines parallel to AB (the first two above AB and the third below) at distances of (i) ·3″, (ii) ·5″, (iii) ·7″. Measure the distance between the two points in which each line meets the locus.

6. Take two points A, B at a distance of ·8″. Plot the locus of a point P which moves so that the difference of the distances of P from A and B is ·4″.

Draw three lines parallel to AB (the first two above AB and the third below) at distances of (i) ·6″, (ii) ·9″, (iii) 1·2″; measure the distance between the two points in which each line meets the locus.

7. Take two points A, B at a distance of 1″. Plot the locus of a point the arithmetic sum of whose distances from A and B is 1″. What result do you obtain?

8. Take two points A, B at a distance of 1″. Plot the locus of a point the arithmetic difference of whose distances from A and B is 1″. What result do you obtain?

9. Take two points A and B at a distance of 1″. Plot the locus of a point P which moves in such a way that AP is always double of BP. Describe exactly the shape and size of this locus.

10. A and B are two points 10 tenth-inches (*i. e.* one inch) apart. Trace the locus of P when $AP \times BP = \text{constant}$.

[When the question is given in this form assume any convenient value for the constant, say 24, as in Question 4; it being then understood that AP and BP are both measured in tenth-inches, and that the product of their measures is 24.

Different values of the constant may give different types of curve. Trace this locus three times taking the constant as (i) 24, (ii) 25, (iii) 36.]

On AB take a point C at a distance of 2 tenth-inches from A. Through C draw a line perpendicular to AB. Measure the chord intercepted by the locus on this line.

122. THEOREM [L. 1].—**The locus of a point which is at a given distance from a given point is [the circumference of] a circle whose centre is the given point and whose radius is the given distance.**

This is a restatement of the definition of a circle and requires no proof. The student must satisfy himself however that a point which is not on the circumference cannot be at the given distance from the given point.

123. THEOREM [L.2].—**The locus of a point which is at a given distance from a given line, on one side of it, is a line drawn parallel to the given line through any point which lies at the given distance from the given line.**

In Fig. 185, AB is the given line, and X the given distance.

DC is $\perp$ to AB and equal to X; also P is any other point on the same side of AB as C, and PQ is the perpendicular distance of P from AB.

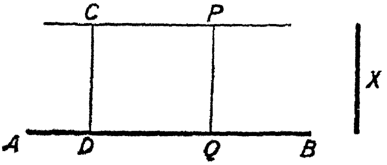

FIG. 185.

(i) *Given* that $PQ = X$, *Required to prove* that P lies on the line through C parallel to AB.

Cons.—Join PC.

Proof.— $\because CD \perp AB$,
and $PQ \perp AB$,

$$\therefore \angle CDQ + \angle PQD = 2\text{R}.$$

Hence (using DQ as trv.) $CD \parallel PQ$. P.2.

In quad. $CDQP$, $\because CD \parallel PQ$
and $CD = PQ$,
$\therefore CP \parallel DQ$. Pm.2

Hence P lies on a line through $C \parallel AB$, and (Playfair's Axiom) there is only one such line.

(ii) † *Given* that P lies on the line through C parallel to AB, *Required to prove* that $PQ = X$.

Proof.— $\because CD$ and PQ are both $\perp AB$,

$$\therefore \angle CDQ + \angle PQD = 2\text{R}.$$

Hence (using DQ as trv.) $CD \parallel PQ$. P.2.

But $CP \parallel DQ$ (*Hyp.*), $\therefore CPQD$ is $\parallel^{\text{m}}$,
$\therefore PQ = CD$. Pm. 1.
Hence $PQ = X$.

COROLLARY.—**The locus of a point which is at a given distance from a given straight line consists of two lines each parallel to the given line one on each side of the line.**

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† This part of the argument is usually omitted, though logically necessary.

124. On the intersection of two loci.—

Ex.—In Fig. 186 AB is a given line and C a given point Find a point which is 1″ from the point C and ·4″ from the line AB.

With centre C and radius 1″ describe ⊙ HKL.
All points on this ⊙ *are* 1″ *from* C.

Draw two lines DE, FG each ∥ AB and ·3″ from AB.
All points on DE *or* FG *are* 3″ *from* AB.

But the points H, K, L, M lie on ⊙ HKL, and also lie on DE or FG.
Hence each of these four points H, K, L, M is 1″ from C and ·3″ from AB.

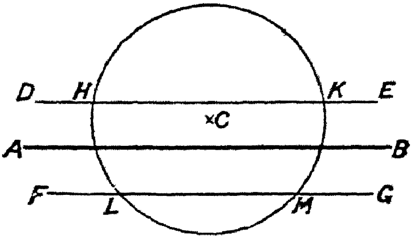

FIG. 186.

The preceding example affords a good illustration of the method of **intersecting loci.**

It is required to find a point which satisfies two conditions, viz. (i) the point is to be 1″ from C, and (ii) the point is to be ·3″ from AB.

Each condition gives a locus, viz. (i) gives the ⊙ HKL, and (ii) gives the pair of lines DE, FG.

All points where **the first locus intersects the second** satisfy both conditions.

EXERCISES XLII (Practical Geometry).

1. Work *Introductory Course*, Ques. 14, Exercises XI.
2. Work *Introductory Course*, Ques. 20, Exercises XI.
3. Work *Introductory Course*, Ques. 16, Exercises XI.
4. Work *Introductory Course*, Ques. 18, Exercises XI.
5. Draw two lines OX, OY at right angles. Find, by the method of intersecting loci, a point P within $\angle XOY$ whose distance from O is 10, and the product of whose distances from OX and OY is 45—all distances being measured in *tenth-inches*. Measure the distances of P from OX and OY.
6. Draw two lines OX, OY at right angles. Find, by the method of intersecting loci, a point P within $\angle XOY$ such that the sum of the distances of P from OX and OY is 12, and the product of these distances is 24—all distances being measured in *tenth-inches*. Measure the distances of P from OX and OY.

125. THEOREM [L.3].—**The locus of a point which is equidistant from two given points is the straight line which bisects at right angles the line joining the two given points.**

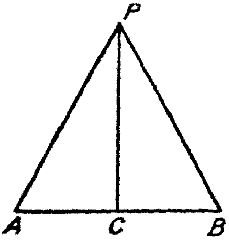

FIG. 187.

In Fig. 187 A, B are the two given points, and C is the middle point of AB.

Given that P is any other point such that $PA = PB$,

Required to prove that P lies on the line at right angles to AB through C.

Cons.—Join PC, PA, PB.

Proof.—

$$\text{In } \triangle\text{s } PCA,\ PCB,\ \because \left\{\begin{matrix} PC = PC \\ CA = CB \\ PA = PB \end{matrix}\right\} \therefore \angle PCA = \angle PCB. \qquad \text{C.3.}$$

But these are adj. angles; $\therefore PC \perp AB$.

Hence P lies on the line at right angles to AB through C.

The student should work for himself the proof of the statement that *any point* on this line is equidistant from A and B by C.1. He must also remember that the line is to be supposed produced indefinitely *both* ways, *i.e.* above and below AB.

126. THEOREM [L.4].—**The locus of a point which is equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the given lines.**

(i) In Fig. 188 AOB, COD are the two intersecting straight lines, and P is a point in $\angle DOB$.

Given that $PM = PN$, and PM, PN, are $\perp$ to OB, OD,

Required to prove that P lies on the bisector of the angle DOB.

Cons.—Join PO.

Proof.—In $\triangle$s PNO, PMO,

$$\because \left\{ \begin{array}{l} OP = OP \\ PN = PM \\ \angle PNO = \angle PMO = R \end{array} \right\} \therefore \angle PON = \angle POM. \quad \text{C.4.}$$

Hence P lies on the bisector of $\angle BOD$.

(ii) If the point P lies in $\angle AOC$ the locus is the bisector of $\angle AOC$; since vertically opposite angles are equal (A.2), this bisector is the line OP produced backwards.

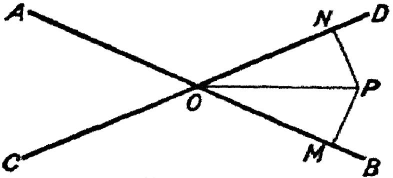

FIG. 188.

Similarly if the point lies in $\angle AOD$ or $\angle BOC$ the locus is the bisector of $\angle AOD$ or $\angle BOC$.

Hence the complete locus is the pair of lines.

The student must prove for himself by C.2 that any point on either of these bisectors is equidistant from AB and CD.

Note that these bisectors are at right angles. See Ques. 12, p. 99.

127. DEFINITIONS.—(i) **Three or more straight lines are said to be *concurrent* if they all pass through one point.**

Thus in Fig. 188 the lines AB, CD, OP are concurrent, since they all pass through O.

(ii) **Three or more points are said to be *collinear* if they all lie on one straight line.**

Thus in Fig. 188 the points A, B, M, O are collinear.

EXERCISES XLIII.

RIDERS.

1. Show how to find a point which shall be at given distances from two given points. How many such points can be found?

2. Show how to find a point which shall be at given distances from two given intersecting straight lines. How many such points can be found?

3. Show how to find a point on a given line (X) which is equidistant from two given points (A, B). When is this impossible?

4. Show how to find a point on a given line which is equidistant from two given intersecting lines. How many such points can be found?

5. Find a point which is equidistant from three given points. When is this impossible?

6. Find a point which is equidistant from three given straight lines. When is this impossible?

7. The straight lines drawn at right angles to the sides of a triangle, from their middle points, are concurrent. L.3.

8. The straight lines which bisect the angles of a triangle are concurrent. L.4.

9. If the sides AB, AC of a triangle ABC be produced to D and E, the bisectors of the angles BAC, CBD, BCE are concurrent. L.4.

10. Find the locus of the vertex of an isosceles triangle drawn on a given base; hence construct an isosceles triangle on a given base having its vertex on a given straight line.

11. Find the locus of the points of intersection of two equal circles drawn with given centres.

12. Find the locus of a point which is equidistant from two given parallel straight lines.

13. The line AB is bisected at C; find the locus of a point at which AC and BC subtend equal angles. C.1, C.5.

PROBLEMS IN PRACTICAL GEOMETRY.

14. Construct a $\triangle$ ABC such that $AB=2''$, $BC=2{\cdot}5''$, $CA=1{\cdot}5''$. Find a point on AB which is equidistant from B and C, and measure its distance from either.

15. Construct a $\triangle$ ABC such that $AB=3$ cms., $BC=4$ cms., $CA=4$ cms. Find a point which is equidistant from A, B and C, and measure its distance from any one of these points.

16. Draw a triangle ABC, such that $AB=3$ cms., $BC=4$ cms., $CA=5$ cms. Find two points, one in CB and one in CB produced, which are equidistant from AB and AC. Measure the distance of each point from AC.

17. Draw a triangle ABC as in question 16. Find a point within the triangle equidistant from all three sides. Measure its distance from any side.

18. Draw a quadrilateral $ABCD$ in which $AB=4$ cms., $BC=4$ cms., $AD=5{\cdot}2$ cms., $\angle A=90°$, and $\angle B=60°$. Find a point P such that $PA=PB$ and $PC=PD$. Measure PC.

19. Draw an acute-angled triangle on a base $2''$, of altitude $1''$, having one side $1{\cdot}3''$. Measure the other side.

20. Draw an obtuse-angled triangle, having the measurements given in Question 19. Measure the other side.

21. Draw a right angle XOY. Plot the locus of a point within the right angle whose distance from OX is three times the distance from OY. Take a point P on OX at a distance $1''$ from O. Measure the shortest distance from P to the locus.

22. Draw an angle POQ of 70°. Plot the locus of a point within the angle which is twice as far from OP as from OQ. Take a point P on OX at a distance 2″ from O. Measure the shortest distance from P to the locus.

23. Draw a line AB of length ·6″. Through B draw CD perpendicular to AB. Plot the locus of a point which moves so that its distance from the point A is always equal to its distance from the line CD. Hence find a point P whose distance from CD is equal to its distance from the point A, and double of its distance from the line AB. Measure AP.

24. Draw a right angle XOY. Plot the locus of a point which moves within the right angle in such a way that its distance from OX always exceeds its distance from OY by a constant. Describe the locus.

25. Draw a right angle POQ. Plot the locus of a point which moves within the right angle in such a way that its distance from OP always exceeds half its distance from OQ by a constant. Describe the locus.

26. Draw an angle HOK of 60°. P is a point within the angle, and PL, PM are the perpendiculars on OH, OK respectively. Plot the locus if P moves so that $PL + 2\,PM = 2''$. Find a point on the locus equidistant from its extremities and measure the distances of this point from OH and OK.

27. P is a point within a right angle XOY, and PM, PN are the perpendiculars on OX, OY respectively. Plot the locus when P moves so that $PM = \frac{1}{10}\,PN^2$, where PN^2 has its usual algebraic meaning, and all distances are measured in *tenth-inches*. Hence find a position in which $PM = \frac{1}{10}\,PN^2$ and $PM + PN = 10$; measure PM, PN.

28. A and B are two fixed points 1·5″ apart. Plot the locus of a point P which moves so that $PA = 2PB$. Draw a line parallel to AB at a distance of ·5″. Measure the length intercepted by the locus on this line.

29. A and B are two fixed points 2″ apart. Plot the locus of a point which moves so that $PA + 2PB = 4''$. Draw lines perpendicular to AB, (i) through B, (ii) bisecting AB. Measure the length intercepted by the locus on each of these lines.

30. Take a line AB; from a point O in AB draw a line OC perpendicular to AB and of length ·9″. P is a moving point, and PM the perpendicular distance of P from AB. Plot the locus of P if $CP = \frac{4}{5}\,PM$. Hence find two positions for P in which $CP = \frac{4}{5}\,PM$, and $PN = 1''$; where PN is the perpendicular on OC. Measure PM in each case.

31. Take a line AB; from a point O in AB draw a line OC perpendicular to AB and of length ·5″. P is a moving point and PM the perpendicular distance of P from AB. Plot the locus of P if $CP = \frac{3}{2}\,PM$. (Note that there will be a portion of the locus on each side of AB.) Hence find two positions for P in which $CP = \frac{3}{2}\,PM$, and $PN = 1''$: where PN is the perpendicular on OC. Measure PM in each case.

CHAPTER VII.

AREAS.

128. **Squared Paper.**—Draw two perpendicular lines XOX', YOY', (say each about two inches long and roughly bisecting each other at the point O). Starting from O mark off along OX lengths of one tenth-inch; do the same along OY, OX', OY'. Through each point of division marked on XOX' draw a line parallel to YOY', and through each point marked on YOY' draw a line parallel to XOX'.

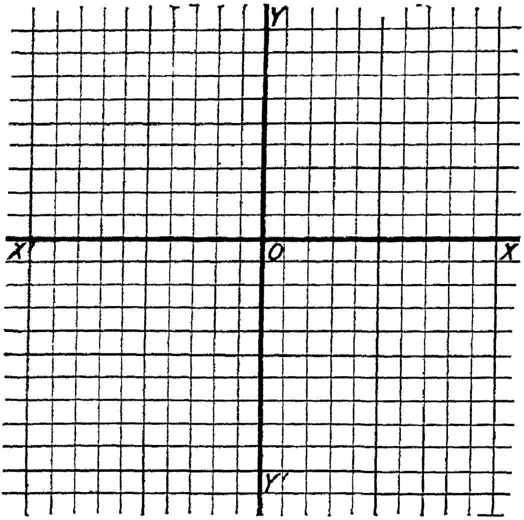

FIG. 189.

The result is a figure entirely made up of a series of equal squares, each side of which measures one tenth-inch.

The formal proof of this is given in the course of the Theorem Ar. 1. It is easily worked as a rider, and depends on P. 3. and Pm. 1.

Paper ruled into squares in this manner is called **squared paper**, and is very useful for many purposes in Practical Geometry. It is called *tenth-inch squared paper* (or *millimetre squared paper*), if the side of each square is one tenth-inch (or one millimetre).

129. **Areas.**—DEFINITION: **The *area* of a figure is the measure of its surface.**

Thus the area of the rectangle *ABCD* (Fig. 191) is the amount of surface contained within the four sides of the rectangle.

If two figures are congruent their areas are equal.

The two triangles *ABC, DEF* (Fig. 190) are congruent, *i.e.* the one triangle can be made to coincide with the other: it follows that the surface contained within the three sides of the triangle *ABC* will coincide with the surface contained within the three sides of the triangle *DEF*; *i.e.* the triangles are equal in area.

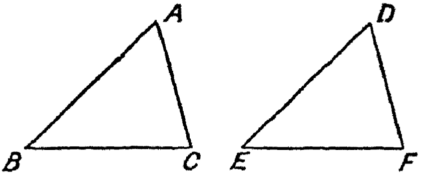

FIG. 190.

Similarly in squared paper, any two squares are equal in area; for squares on equal sides are congruent. Pm.7.

Henceforth the student must be careful to remember that:—

All propositions on congruence assert equality of area.

Two figures which are not congruent may be equal in area.

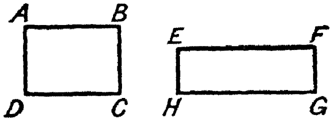

FIG. 191.

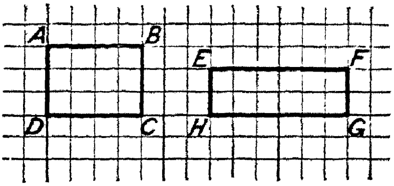

FIG. 192.

The rectangles *ABCD, EFGH* (Fig. 191) are not congruent, and it cannot be determined whether they are equal or not by applying one to the other. But when they are both placed on squared paper (as in Fig. 192) it is seen that each covers exactly 12 squares. Since the squares are all equal it follows that *ABCD* and *EFGH* are equal in area.

The sign $\equiv$ applied to rectilinea figures denotes congruence *and* equality of area; the sign $=$ denotes equality of area *without* congruence. Thus in Fig. 190 $\triangle ABC \equiv \triangle DEF$, but in Fig. 191 rect. $ABCD =$ rect. $EFGH$.

EXERCISES XLIV (Calculations).

1. Copy Fig. 193 on tenth-inch squared paper; and mark on your copy the number of tenth-inch squares contained in each rectangle.

2. Are any two of the rectangles in Fig. 193 equal in area?

3. Is any one of the rectangles in Fig. 193 equal in area to the sum of two others?

4. How many times is the area of the rectangle *C* contained (i) in the area of the rectangle *B*, and (ii) in the area of the rectangle *E*?

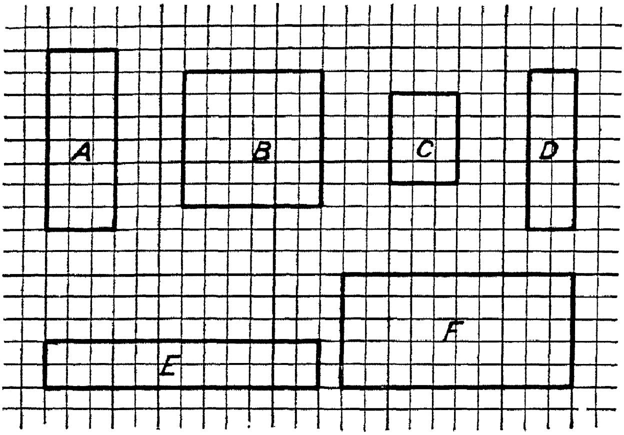

FIG. 193.

5. When rectangles are drawn on squared paper as in Fig. 193, find a rule for calculating the number of squares contained in the rectangle, given the number of tenth-inches contained in each of two adjacent sides of the rectangle. That is :—
If two adjacent sides of the rectangle measure p tenth-inches and q tenth-inches respectively, how many squares does the rectangle contain?

130. **On Units of Area.**—In order to measure any given length we determine how many times this length contains some "unit" of length such as the inch, the centimetre, the foot, etc. Similarly in order to measure any given area we must determine how many times this area contains some "unit" of area.

It is found convenient in any system of measurement to take as unit of area *a square whose side is the unit of length.*

DEFINITIONS.—**A** *square inch* **is the area of a square whose side measures 1 inch.** (Fig. 194.)

A square centimetre is the area of a square whose side measures 1 centimetre. (Fig. 194.)

Other common units of area are the *square foot*, *square yard*, *square mile*, *square metre*, etc.

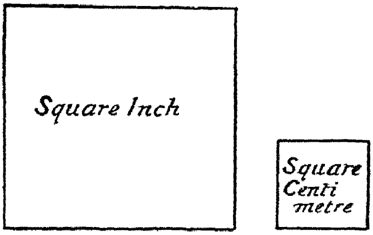

FIG. 194.

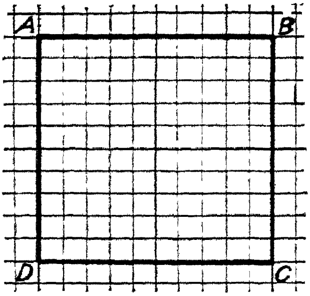

FIG. 195.

In Fig. 195 *ABCD* is a square inch drawn on tenth-inch squared paper. *ABCD* contains 10 rows of squares, also each row contains 10 squares; hence *ABCD* contains 100 squares.

Thus *on tenth-inch squared paper the area of each square is* $\frac{1}{100}$ *sq. inch.*

Similarly by drawing a square centimetre on millimetre squared paper it can be shown that 1 *square centimetre contains* 100 *square millimetres.*

131. **On the Measurement of Areas.**—The area of a geometrical figure can often be determined by drawing the figure on squared paper.

If the figure is entirely made up of whole squares the area is determined by counting the squares.

Thus in Fig. 192 each rectangle contains 12 squares; hence the area of each rectangle is $12 \times \frac{1}{100}$ sq. inch, *i. e.* ·12 sq. inch.

If, however, the figure is not entirely made up of whole squares (as the circle in Fig. 196) some attempt must be made to estimate the area of the "broken squares" included within the figure. The following rule works on a principle of averages, and usually gives approximately accurate results †:—

RULE.—*Portions of squares greater than one-half should be counted as whole squares; portions less than one-half should be counted as nothing; portions exactly one-half should be counted as such.*

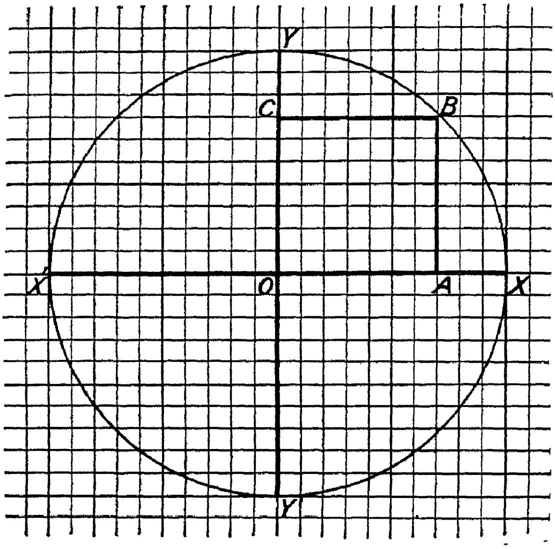

FIG. 196.

Example.—*Determine the area of a circle of radius* 1 *inch.*

Using tenth-inch squared paper, draw a circle with centre *O* and radius 1 inch (Fig. 196).

---

† This rule is of no value if the number of squares included within the figure is small—say less than twenty-five. If the number of squares is large—say greater than a hundred—the error is seldom greater than two squares; but the rule is more reliable for *regular curves* than for any other type of figure. It is obviously very easy to draw a rectangle for which the rule completely fails.

In this case it may be assumed that the lines *XOX′*, *YOY′* divide the circle into four equal areas; accordingly it is only necessary to count the squares in one of these areas, say the quadrant[†] *XOY.*

The "broken squares" occur round the circumference and are to be counted 1, 0, or $\frac{1}{2}$, according as they are greater than, less than, or equal to half a square.

To count the squares:—

$$\begin{array}{ll}
\text{No. of squares in } OABC \text{ is } (7 \times 7) \text{ or} & 49 \\
\text{No. of squares in the three rows above } BC \text{ are respectively} & \left\{\begin{array}{l} 6\frac{1}{2} \\ 5 \\ 3 \end{array}\right. \\
\text{No. of squares in the three columns to the right of } AB \text{ are respectively} & \left\{\begin{array}{l} 6\frac{1}{2} \\ 5 \\ 3 \end{array}\right. \\
\hline
\text{Total no. of squares in quadrant } XOY = & 78
\end{array}$$

$$\begin{aligned}
\text{Thus area of circle} &= 4 \times 78 \text{ squares} \\
&= 4 \times 78 \times \tfrac{1}{100} \text{ sq. inches} \\
&= \mathbf{3{\cdot}12 \text{ sq. inches.}}
\end{aligned}$$

NOTE.—The *correct* answer (obtained by other methods) is 3·14 sq. inches.

EXERCISES XLV.

[In Questions 2–9 use tenth-inch squared paper.]

1. What is the area of each of the rectangles in Fig. 193?

2. Find the area of an equilateral triangle of side 2 inches.

3. Find the area of a triangle whose sides are 2″, 2·5″, 3″.

4. Find the area of a circle of radius ·8″.

5. Find the area of a circle of radius 1·2″.

6. Find the area of a quadrant of a circle of radius 3″, and deduce the area of the whole circle.

7. In each of the three preceding questions find (correct to three significant figures) the ratio of the area of the circle to the area of a square whose side is the radius of the circle. What do you infer?

8. A chord divides a circle into two parts. Find their areas if the radius is 1·6″ and the chord is ·8″ from the centre.

9. Find the area of the smaller portion cut off from a circle of radius 3″ by a chord of length 3″.

10. Draw a rough sketch to represent a square yard divided into square feet. How many square feet are there in a square yard?

11. How many square inches are there in a square foot?

12. How many square furlongs are there in a square mile?

[†] A quadrant is the part of a circle contained between two perpendicular radii.

132. Area of a rectangle.—The theorems in the remainder of this chapter show how to calculate the areas of various rectilinear figures. The easiest case, as the student will already have noticed, is the area of a rectangle.

In order to construct a rectangle it is only necessary to know the lengths of two adjacent sides. For the included angle is a right angle, and the remaining sides must be drawn parallel to the given sides. In other words—a rectangle is *completely determined* when the lengths of two adjacent sides are given.

The rectangle in which two adjacent sides are equal to two given lines X and Y, is described as the **rectangle contained by the lines** X, Y—a phrase which is usually written in the abbreviated form **rect.** (X, Y).

133*. Commensurable and Incommensurable Quantities. In the next proposition it is proved that the area of a rectangle can be determined if we know the lengths of the sides. The question at once occurs whether we can always express the sides *exactly* in terms of the unit of length; *i. e.* whether the measure of each side in terms of the unit of length is an integer or a fraction or cannot be expressed *exactly* either as an integer or fraction. In this connection notice that a finite or a recurring decimal can always be reduced to an *exactly equivalent* vulgar fraction.

DEFINITION.—**Two quantities are said to be *commensurable* if it is possible to find a third quantity which is contained in each of them an exact number of times.**

Thus two lines whose lengths are 1′ 6″ and 2′ 6″ are commensurable, for a length of one inch is contained 18 times in the first and 30 times in the second (or a length of 6 inches is contained 3 times in the first and 5 times in the second). So again two lines whose lengths are 1·37″ and 2·6″ are commensurable; for a length of $\frac{1}{100}$ inch is contained 137 times in the first and 260 times in the second.

DEFINITION.—**Quantities which are not commensurable are called** *incommensurable.*

We do not deal with incommensurable quantities at all in this part of the work.

We may take $\sqrt{2}$ and 1 as the simplest example of two incommensurable quantities, and these actually occur very frequently in Geometry; for example, the diagonal of a square whose side is 1″ may be proved to be $\sqrt{2}$″.

Incommensurable quantities introduce considerable difficulty into the logical theory of geometry, but they do not cause any difficulty in practical work; for we can always replace incommensurable quantities by commensurable quantities which are practically equal to them—in fact we can always obtain commensurable quantities which are equal to given incommensurable quantities to *any required degree of accuracy* (short of absolute theoretical accuracy).

Thus we can find $\sqrt{2}$ as a decimal *correct to any required decimal place.* Correct to the 7th decimal place the result is 1·4142135. Hence, (in any calculation,) instead of the incommensurable quantities 1 and $\sqrt{2}$, we may use the commensurable quantities 1 and 1·4, or 1 and 1·41, or 1 and 1·414, etc.—according to the degree of accuracy required in the result.

It may be noted in passing that probably *no* two lengths occurring in nature are really commensurable.

If any length is commensurable with the unit of length, *its measure* (in terms of that unit) *is either an integer or a simple vulgar fraction.*

For the given length and the unit of length both contain some third length an exact number of times—say p times and q times respectively (where p and q are integers). Hence the given length $= p/q \times$ unit of length; and therefore the measure of the given length is p/q, which is either an integer or a simple vulgar fraction.

134. *Prel. Ex.*—(i) A squad of boys is marching in fours. There are twelve ranks; how many boys are there?

(ii) Two adjacent sides of a rectangle measure 3 yds. and 7 yds. respectively. Draw a rough sketch representing this rectangle divided into square yards, and hence determine its area.

(iii) Draw a square inch and divide it into rectangles each $\frac{1}{3}$ inch long and $\frac{1}{5}$ inch high.

What is the area of each rectangle?

(iv) Draw a rectangle $\frac{7}{3}$ inches long and $\frac{4}{5}$ inch high. Divide it into rectangles each $\frac{1}{3}$ inch long and $\frac{1}{5}$ inch high.

Hence and from (iii) show that the area of this rectangle in square inches is $\frac{7}{3} \times \frac{4}{5}$ *i. e.* $\frac{28}{15}$.

THEOREM [Ar. 1].†—**The area of a rectangle is the product of the lengths of two adjacent sides.**

This theorem is more correctly stated as follows :—

The measure of the area of any rectangle in any units of area is the product of the measures of two adjacent sides in the corresponding units of length.

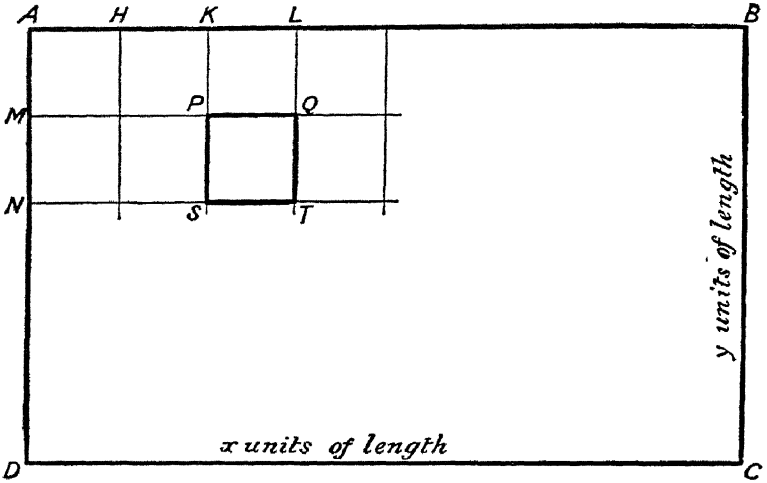

FIG. 197.

CASE I.—When the measure of each side in terms of the unit of length is an integer.

In the rect. *ABCD* (Fig. 197) :—

Given that *AB* contains x units of lengths ‡ (*AH, HK, KL,* etc.), and *AD* contains y units of length (*AM, MN,* etc.)

Required to prove that rect. *ABCD* contains xy units of area.

Cons.—From *H, K, L,* . . . . draw lines parallel to *AD,* terminated by the side *DC.*§

From *M, N,* . . . draw lines parallel to *AB,* terminated by the side *BC.*§

---

† The first proposition in Areas.

‡ The actual unit of length used in Fig. 197 is the centimetre.

§ In Fig. 197 these lines are not completed. This is in order to avoid the appearance of arguing from a special case.

Proof.—(i) The lines drawn parallel to AD are all parallel to one another. P.3.

The lines drawn parallel to AB are all parallel to one another. P.3.

$$\left.\begin{aligned}&\text{Hence } PQLK \text{ is } \|^{\text{m}}, \text{ whence } PQ=KL;\\&STLK \text{ is } \|^{\text{m}}, \text{ whence } ST=KL;\\&PSNM \text{ is } \|^{\text{m}}, \text{ whence } PS=MN;\\&QTNM \text{ is } \|^{\text{m}}, \text{ whence } QT=MN.\end{aligned}\right\}\quad \text{Pm.1.}$$

Thus each side of quad. $PQTS$ is of unit length.

$\because$ PQ, PS are respectively parallel to and in the same sense with AB, AD,

$\therefore$ $\angle SPQ = \angle A = \text{R}$. P.4.

Hence $PQTS$ is a square (*Def.*) and is equal to the unit of area (*Def.*).

Similarly any other of the quadrilaterals into which the rectangle is divided is a square and is equal to the unit of area.

(ii) $\because$ AB contains x units of length,
$\therefore$ rect. $ABCD$ is divided into x columns of squares.
$\because$ AD contains y units of length,
$\therefore$ each column contains y squares.
Hence rect. $ABCD$ is divided into $x \times y$ squares.

But each square is equal to the unit of area.
Hence area of rect. $ABCD = x \times y$ (i. e. xy) units.

Case II.*—When the measure of each side in terms of the unit of length is a simple vulgar fraction.†

In rect. $ABCD$ (Fig. 198) :—

Given that
AB contains p/q units of length, and
AD contains r/s units of length.

Required to prove that
$ABCD$ contains pr/qs units of area.

---

† Note that this includes the case of decimals (both terminating and recurring) as these can always be reduced to vulgar fractions. Also the product of two decimal fractions is equal to the product of the equivalent vulgar fractions.

Cons.—Let *EFGH* be the unit of area, *i. e.* a square in which each side is the unit of length.†

Divide *AB* into p parts; then each contains $1/q$ unit of length. Divide *EF* into q parts; then each contains $1/q$ unit of length.

Divide *AD* into r parts; then each contains $1/s$ unit of length. Divide *EH* into s parts; then each contains $1/s$ unit of length.

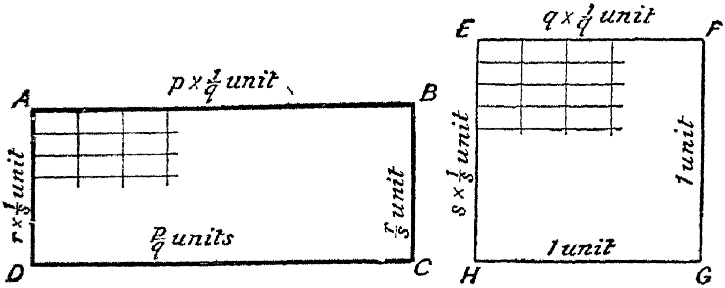
A
B
C
D
$p \times \frac{1}{q}$ unit
$r \times \frac{1}{s}$ unit
$\frac{p}{q}$ units
$\frac{r}{s}$ unit
E
F
G
H
$q \times \frac{1}{q}$ unit
$s \times \frac{1}{s}$ unit
1 unit
1 unit

FIG. 198.

Divide up both figures by lines drawn through the points of division of each side || to the adjacent sides.

Proof.—Both figures are now divided into a series of equal rectangles each of which measures $1/q$ unit in breadth and $1/s$ unit in height. ‡

∵ *EFGH* contains q columns of rectangles, and each column contains s rectangles,

∴ *EFGH* contains qs rectangles.

But *EFGH* is the unit of area;

Hence each rectangle contains $1/qs$ unit of area.

∵ *ABCD* contains p columns of rectangles, and each column contains r rectangles,

---

† The actual unit of length used in Fig. 198 is the inch.

‡ The proof of this obvious statement is omitted for the sake of brevity. It involves three steps—(i) to show that all the angles of each quadrilateral are right angles [P.4], and hence each quadrilateral is a rectangle (*Def.*); (ii) to show that the two adjacent sides of each rectangle measure $1/q$ and $1/s$ unit of length [P.3; Pm.1]; (iii) to show that the rectangles are all congruent [Pm. 7].

ABCD contains pr rectangles, *i. e.* $pr \times 1/qs$ units of area.

Hence *ABCD* contains pr/qs units of area.

CASE III.*—When one or both of the sides is not commensurable with the unit of length.

The consideration of this case is exceedingly difficult, and must be omitted for the present. It should, however, be noticed:—

(i) That no practical difficulty arises from this omission as it is always possible to express any length as a terminating decimal to *any* required degree of accuracy, and hence to calculate the area of any rectangle to any required degree of accuracy.

(ii) That this omission will not constitute a defect in the course of logical argument, as it will be found that this proposition is not quoted in the proof of subsequent propositions, but only in the proofs of formulæ to be used in calculations.

COROLLARY.—**If the area of a rectangle is divided by the length of one side, the result gives the length of an adjacent side—provided that area and length are expressed in corresponding units.**

EXERCISES XLVI (Calculations).

1. Find the area of a rectangle whose length and breadth are respectively:—(i) 3·5″, 2·1″; (ii) 4·7 cms., 1·7 cms.; (iii) ·001″, ·0001″; (iv) 2·35 cms., ·41 cms.

2. Find the breadth of a rectangle whose length and area are respectively:—(i) 25″, 100 sq. in.; (ii) 5″, 1 sq. in.; (iii) 5″, ·01 sq. in.; (iv) 20 cms., 20 sq. cms.

3. Find the length of a rectangle whose breadth and area are respectively:—(i) 2″, 10 sq. in.; (ii) ·2″, 10 sq. in.; (iii) ·4 cm., 1·96 sq. cms.; (iv) 1·28 cms., 5 sq. cms.

4. The side of a square lawn measures 25 yards; find its area in square yards.

5. A county is in the shape of a rectangle 37 miles long and 17 miles broad. How many square miles of land does it contain?

6. A garden is to be 25 feet broad and to contain an area of 4,000 square feet. Find its length.

7. A road connecting two towns is 42 feet broad and 1 square mile in area. Find in miles the distance between the two towns.

8. A rectangular lawn measures 20 yards long and 10 yards wide, and is surrounded by a gravel path 1 yard wide. Find the area of this path in square yards.

9. Find the area of the following figures (if drawn to the dimensions marked) by dividing them up into rectangles :—

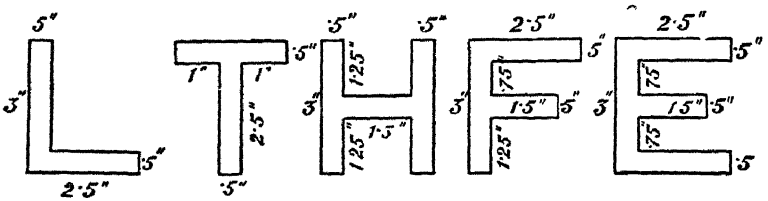

FIG. 199.

10. Draw a figure to show that the area of a rectangle whose adjacent sides are $\frac{1}{2}''$ and $1\frac{2}{3}''$ is $\frac{5}{6}$ square inch.

11. Show by the use of tenth-inch squared paper that if the sides of a rectangle measure 1·2″ and ·7″, its area is ·84 square inch.

12*. Prove that if the sides of a rectangle measure 2·53″ and ·7″, then its area is 2·53 × ·7 square inches. [Use a new unit of length equal to $\frac{1}{100}$ inch, and show that the corresponding unit of area is $\frac{1}{10000}$ square inch.]

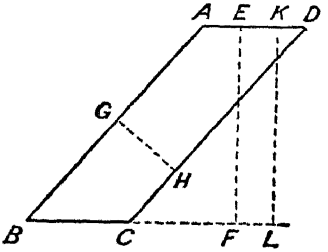

FIG. 200.

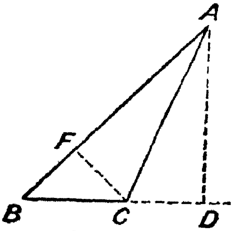

FIG. 201.

135. DEFINITION.—**The** *altitude* **(or** *height*) **of a parallelogram is the perpendicular distance between the base and the opposite side.**

Thus, in ||ᵐ *ABCD* (Fig. 200), if *BC* is regarded as the base, *EF* (or *KL*, which is equal to *EF* by Pm. 1) is the altitude ; if *AD* is regarded as the base, *EF* (or *KL*) is again the altitude ; if either *AB* or *CD* is regarded as the base *GH* is the altitude.

DEFINITION.—**The** *altitude* **(or** *height*) **of a triangle is the perpendicular distance of the vertex from the base.**

Thus, in △ *ABC* (Fig. 201), if *BC* is regarded as the base *AD* is the altitude ; and if *AB* is regarded as the base, *CF* is the altitude.

EXERCISES XLVII (Practical).

1. Copy Fig. 202 roughly to the dimensions given; *ABCD* and *EFGH* are *parallelograms* lying between the same parallels (*X* and *Y*). Draw *DK* ∥ *HE*, and *GL* ∥ *CB*.

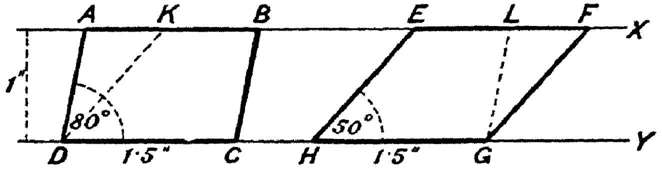

FIG. 202.

Using tracing paper show that the two parts into which ∥^{m} *EFGH* is divided by the line *GL* are respectively congruent to the two parts of ∥^{m} *ABCD.* Shade the larger part of each ∥^{m} by lines ∥ to *X*, and the smaller part by lines ∥ to *AD*, so as to make the congruence of the parts more obvious to the eye.

[Cut out ∥^{m} *EFGH* with scissors or a sharp penknife, and divide it into two parts by a cut along *LG*. Now paste or pin these two parts on to ∥^{m} *ABCD*, so as to cover this parallelogram exactly.]

2. Copy Fig. 203 roughly to the dimensions given; *ABCD* and *EFGH* are to be *parallelograms* lying between the same parallels; also *HE* is to be *parallel* to *DB*.

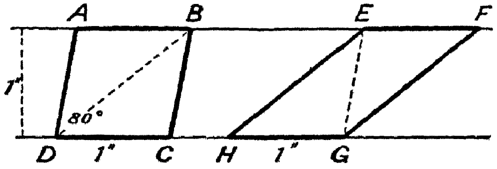

FIG. 203.

Using tracing paper show that the two parts into which ∥^{m} *EFGH* is divided by the diagonal *EG* are respectively congruent to the two parts of ∥^{m} *ABCD.* Shade △*EGH* by lines parallel to *HG*, and △*EFG* by lines parallel to *EG*, and shade the corresponding triangles in ∥^{m} *ABCD* in the same way, so as to make the congruence of the parts more obvious to the eye.

[Cut out the two parts of ∥^{m} *EFGH*, and paste or pin them on to ∥^{m} *ABCD*, so as to cover this parallelogram exactly.]

3. Copy Fig. 204 accurately to the dimensions given, from the following directions :—$ABCD$ and $EFGH$ are two parallelograms lying between the same parallels ; $DK \parallel HE$, $GL \parallel CB$; $DM = MN = CK$, $NS \parallel MR \parallel DK$; $GP = PQ = HL$, $QY \parallel PT \parallel GL$.

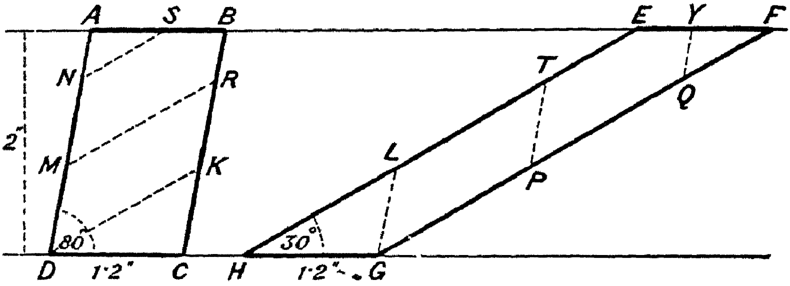

FIG. 204.

Using tracing paper show that the four parts of $\|^{m}$ $EFGH$ are respectively congruent to the four parts of $\|^{m}$ $ABCD$. Shade the various parts of the two figures in such a way as to make the congruence of the various parts more obvious to the eye.

[Cut out the four parts of $\|^{m}$ $EFGH$ and paste or pin them on to $\|^{m}$ $ABCD$ so as to cover this parallelogram exactly.]

4. What do you infer from Questions 1–3 ?

136. *Prel. Ex.*—In Figs. 206, 207, $ABCD$ and $EBCF$ are parallelograms: prove that $\triangle DCF \equiv \triangle ABE$. What figure remains when $\triangle DCF$ is cut away from quad. $ABCF$? What figure remains when $\triangle ABE$ is cut away from quad. $ABCF$? What can you infer ?

THEOREM [Ar. 2].—**Parallelograms on the same base and between the same parallels are equal in area.**

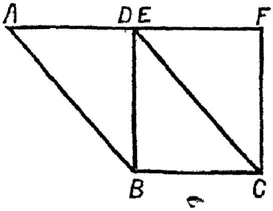

FIG. 205.

In Figs. 205, 206, 207 :—

Given that $ABCD$, $EBCF$ are $\|^{ms}$ on the same base BC and between the same $\|^{s}$ BC and AF,

Required to prove that

$$\|^{m} ABCD = \|^{m} EBCF.$$

Proof.—CASE I.: When the points D, E coincide (Fig. 205).

Each of the $\|^{ms}$ $ABCD$, $EBCF$ is double of $\triangle DBC$. Pm. 1.

$$\therefore \|^{m} ABCD = \|^{m} EBCF.$$

CASE II.—When the points D and E do not coincide (Figs. 206 and 207).

(1) In $\|^{\text{m}}$ $ABCD$, $AD=BC$; in $\|^{\text{m}}$ $EBCF$, $EF=BC$. Pm.1.
Hence $AD=EF$.

In Fig. 206 to each add DE; in Fig. 207 from each take DE; thus in either figure $AE=DF$.

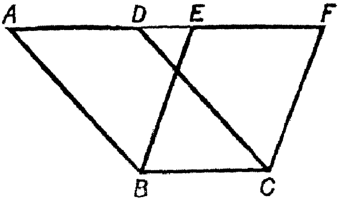

FIG. 206.

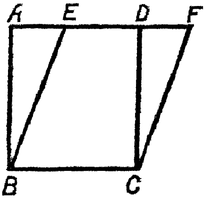

FIG. 207.

(2) In $\|^{\text{ms}}$ $ABCD$, $EBCF$; $AB=DC$, $EB=FC$. Pm.1.

$$\text{In } \triangle\text{s } ABE,\ DCF,\ \because \left\{\begin{array}{l} AE=DF \\ AB=DC \\ EB=FC \end{array}\right\} \therefore \triangle ABE \equiv \triangle DCF. \quad \text{C.3.}$$

(3) When $\triangle DCF$ is taken from quad. $ABCF$, the remainder is $\|^{\text{m}}$ $ABCD$. When $\triangle ABE$ is taken from quad. $ABCF$, the remainder is $\|^{\text{m}}$ $EBCF$.

Since equal triangles have been taken from the same quadrilateral, these remainders are equal.

Hence $\|^{\text{m}}\ ABCD = \|^{\text{m}}\ EBCF$.

CoROLLARY.—**The measure of the area of a parallelogram is the product of the measures of its base and altitude.**

If $ABCD$ is the $\|^{\text{m}}$, draw $BE \perp AD$, and $CF \parallel BE$ (Fig. 208).

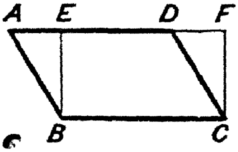

FIG. 208.

Then $BEFC$ is a parallelogram having one angle a right angle, *i. e.* $BEFC$ is a rectangle. Pm.3.

Hence area $BEFC = BC \times BE$ Ar.1.

But $\|^{\text{m}}\ ABCD = \|^{\text{m}}\ BEFC$; Ar.2.

hence area $ABCD = BC \times BE$ = base × altitude.

137. THEOREM [Ar. 3].—**If a parallelogram and a triangle be on the same base and between the same parallels then the area of the parallelogram shall be double that of the triangle.**

In Fig. 209 :—

Given that $\|^{m}$ $ABCD$ and $\triangle EBC$ are on the same base BC and between the same parallels BC, AE,

Required to prove that $\|^{m}$ $ABCD = 2\triangle EBC$.

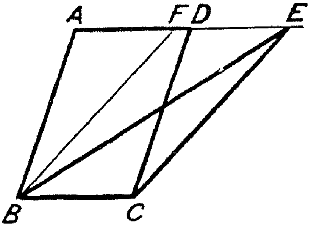

FIG. 209.

Cons.—Draw $BF \parallel CE$ meeting AE at F.

Proof.—$\because BF \parallel CE$ and $BC \parallel FE$, $\therefore FBCE$ is $\|^{m}$.

Hence $FBCE = 2 \triangle EBC$. Pm.1.

$\because \|^{ms}$ $ABCD$, $FBCE$ are on the same base BC and between the same parallels BC, AE,

$\therefore \|^{m} ABCD = \|^{m} FBCE$. Ar.2.

Hence $\|^{m} ABCD = 2 \triangle EBC$.

COROLLARY 1.—**The area of a triangle is half the product of its base and altitude.**

For in Fig. 209 altitude of $\triangle EBC =$ altitude of $\|^{m}$ $ABCD$; and area of $\triangle EBC = \frac{1}{2}$ area of $\|^{m}$ $ABCD$.

COROLLARY 2.—**The area of a trapezium is half the product of the sum of its parallel sides and the perpendicular distance between them.**

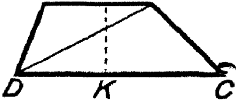

FIG. 210

In Fig. 210, HK is the perpendicular distance between the parallel sides AB, CD.

Then area of $\triangle ABD = \frac{1}{2} AB \times HK$; and area of $\triangle BCD = \frac{1}{2} CD \times HK$. Adding these we obtain the area of trapezium $ABCD$, viz. $\frac{1}{2}(AB + CD) \times HK$.

EXERCISES XLVIII.†

BY CONSTRUCTION AND CALCULATION.

1. Draw a parallelogram in which two adjacent sides measure 2″ and 3″ respectively and include an angle of 65° ; determine its area.

2. Draw a triangle whose sides measure 4, 5 and 6 cms. respectively, and determine its area in sq. cms.

3. Draw a parallelogram in which two adjacent sides measure 2″ and 2·5″ respectively and one diagonal measures 3″. Determine its area.

4. Draw a triangle on a base 5 cms. having base angles of 40° and 80° respectively. Determine its area in square centimetres.

5. Draw the triangles given in Questions 1–3 of Exercises IV (p. 16) and determine the area of each.

6. Draw the parallelograms given in Questions 2 and 3 of Exercises XIII (p. 47) and determine the area of each.

138. THEOREM [Ar. 4].—**Parallelograms on equal bases and between the same parallels are equal in area.**

Given that $ABCD$, $EFGH$ (Fig. 211) are $\|^{ms}$ on equal bases BC, FG and between the same parallels BG, AH,

Required to prove that $\|^{m}\ ABCD = \|^{m}\ EFGH$.

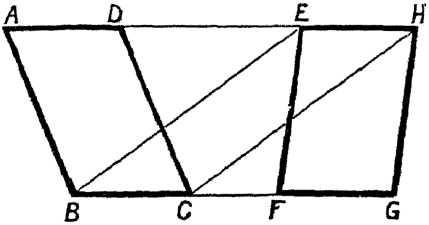

FIG. 211.

Cons.—Join EB, HC.

Proof.—In $\|^{m}\ EFGH$,
$EH = FG$; Pm. 1.
but $BC = FG$,
hence $EH = BC$.

$\because\ EH = BC$ and $EH \parallel BC$, $\therefore\ EBCH$ is $\|^{m}$ Pm. 2.

$\because\ \|^{ms}\ ABCD, EBCH$ lie on the same base BC and between the same parallels BC, AH,

$\therefore\ \|^{m}\ ABCD = \|^{m}\ EBCH$. Ar. 2

$\because\ \|^{ms}\ EFGH, EBCH$ lie on the same base EH and between the same parallels EH, BG,

$\therefore\ \|^{m}\ EFGH = \|^{m}\ EBCH$. Ar. 2.

Hence $\|^{m}\ ABCD = \|^{m}\ EFGH$.

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† Where an area is obtained by a process involving measurement the result should be given to three significant figures only. Full marks may be given where the error does not exceed 1 per cent., and half marks where it does not exceed 3 per cent.

139. THEOREM [Ar. 5].—**Triangles on equal bases and between the same parallels are equal in area.**

In Fig. 212:—*Given* that $\triangle$s ABC, DEF are on equal bases BC, EF and between the same parallels BF, AD,

Required to prove that $\triangle ABC = \triangle DEF$.

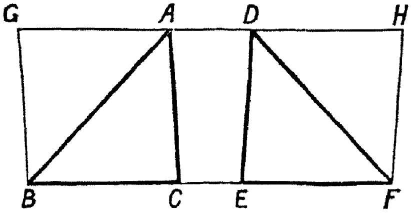

FIG. 212.

Cons.—Draw $BG \parallel CA$ meeting AD produced at G, and $FH \parallel ED$ meeting AD produced at H.

Proof.—$\because BC \parallel GA$, and $BG \parallel CA$, $\therefore GBCA$ is $\parallel^{\text{m}}$.

Hence $\triangle ABC = \frac{1}{2} \parallel^{\text{m}} GBCA$. Pm.1.

Similarly $\triangle DEF = \frac{1}{2} \parallel^{\text{m}} DEFH$. Pm.1.

$\because \parallel^{\text{ms}} GBCA, DEFH$ are on equal bases BC, EF and between the same parallels BF, GH,

$\therefore \parallel^{\text{m}} GBCA = \parallel^{\text{m}} DEFH$. Ar.4.

Hence $\triangle ABC = \triangle DEF$.

COROLLARY 1.—**Triangles on the same base and between the same parallels are equal in area.**

COROLLARY 2.—**A triangle is bisected by each of its medians.**

NOTE.—The theorem is still true if the equal bases lie one on one parallel and one on the other.

140. THEOREM [Ar. 5c].—**Equal triangles on equal bases in the same straight line, and on the same side of it, are between the same parallels.**

In Fig. 213:—*Given* that $\triangle$s ABC, DEF are on equal bases BC, EF in the same straight line, and on the same side of it, and also that $\triangle ABC = \triangle DEF$,

Required to prove that $AD \parallel BF$.

Cons.—Draw $AG \parallel BF$,† meeting ED or ED produced in G. Join GF.

† Do not assume that AG does or does not coincide with AD.

Proof.—$\because$ $\triangle$s *ABC, GEF* are on equal bases *BCEF*, and between the same parallels *BF, AG*,

$\therefore \triangle ABC = \triangle GEF$. Ar.5.

But $\triangle ABC = \triangle DEF$; hence $\triangle GEF = \triangle DEF$.

But this is only possible if *D* and *G* coincide.

Thus *AG* coincides with *AD*.

But $AG \parallel BF$;
hence $AD \parallel BF$.

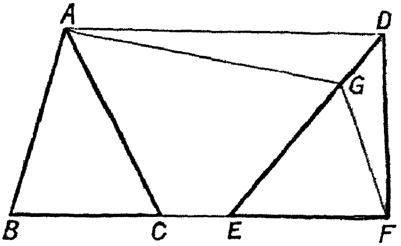

FIG. 213.

COROLLARY 1.—**Equal triangles on the same base and on the same side of it lie between the same parallels.**

COROLLARY 2.—**Triangles which are equal in area and on equal bases have equal altitudes.**

141*. It is an interesting exercise to exhibit the equality in area of the triangles *ABC, DBC* (Fig. 214) by congruence, *i. e.* by cutting them up into pieces which fit one another. This may be done as follows:—

Let *AC, BD* intersect at *O*.

Let *H, K, N, P* be the middle points of *AB, BD, AC, CD*. These are in one straight line, which is $\parallel$ the given parallels. T.6. and Playfair.

Suppose *O* to lie between this line and *AD*.

Through *A* draw $AL \parallel CD$; through *D* draw $DM \parallel AH$.

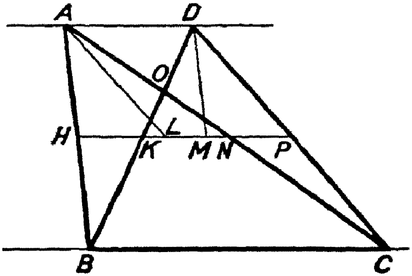

FIG. 214.

Then it will be found that if the triangle *ACB* be cut into the four pieces *AHL, ALN, HKB, BKNC* these pieces can be fitted together to form the $\triangle BDC$; for it is easily proved that $\triangle AHL \equiv \triangle DMP$, $\triangle ALN \equiv \triangle CNP$, $\triangle HKB \equiv \triangle DKM$. Pm.1; P.1*c*.; P.2*c*.; C.2.

The student is recommended to verify this by cutting out any triangle *ABC* in cardboard, dividing this $\triangle$ by *HN* through the middle points of its sides, then dividing again by any cut whatever through *A* and any cut whatever through *B* which divides the side *HN*. The four pieces can now be fitted together to form another triangle which can be placed between the same parallels. He is recommended further (1) to examine the modification required for the above construction when O lies between HP and BC; (2) to construct for himself a similar method of verifying Ar. 2.†

142. THEOREM [L. 5].—**The locus of the vertex of a triangle of given area and standing upon a given base consists of two straight lines parallel to the base.**

In Fig. 215:—

Required to prove that the locus of the vertex of a triangle on base BC, equal in area to some given figure X, consists of two straight lines parallel to BC.

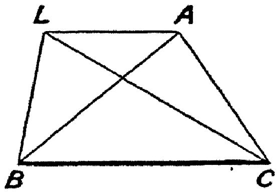

FIG. 215.

Cons.—Let ABC be a triangle on base BC equal in area to X; and let LBC be any other such triangle on the same side of BC. Join LA.

Proof.—$\because$ $\triangle$s LBC, ABC are equal in area and on the same side of the same base BC,

$\therefore LA \parallel BC$. Ar. 5c.

But only one line can be drawn through $A \parallel BC$.
Hence L always lies on the line through $A \parallel BC$.

It is easy to prove conversely that if L be *any* point on this line, then $\triangle LBC = \triangle ABC =$ area X. Ar. 4.

Hence the line through $A \parallel BC$ is part of the required locus.

Similarly it can be shown that there is a second part of the required locus corresponding to triangles drawn on the opposite side of the base BC, and consisting of a second line parallel to BC.

Hence the locus consists of two lines parallel to BC.

NOTE.—By Ar. 5c, Cor. 2, these two lines are equidistant from BC.

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† An interesting set of dissected models in wood by Mr. T. Gosset, for illustrating this and similar propositions, is published by Messrs. G. Philip and Son at 4*s*. See also an article by Mr. H. M. Taylor in the *Messenger of Mathematics*, Oct. 1905.

143. **On Constant Quantities.**—It is obvious from Theorem L.5 that if *ABC* is a triangle (Figure 215), and if we suppose that the points *B* and *C* remain fixed while the vertex *A* travels along the line through *L* parallel to *BC*, then the area of the triangle *ABC* remains the same for all positions of *A*. We may express this by saying that if the vertex of a triangle moves in a line parallel to the base then the area of the triangle is **constant.**

It is easy to see that in the same case the *altitude* of the triangle would also be constant.

In general, whenever by altering a geometrical figure in accordance with certain conditions, we find that some magnitude connected with the figure does not alter, that magnitude is said to be **constant** for these given conditions. Other instances of constant geometrical quantities are given in §168.

144. **On the Solution of Riders.**—As the student has now had some practice in the solving of easy riders, he will be in a position to understand the following hints, and will find them useful in the more difficult work which lies before him.

(1) Commence by drawing a careful figure which represents the **essential points of the rider.** Indicate the details of the hypothesis (such as equal lines, equal angles, etc.) in some way on the figure, and write down the result which it is required to prove.

If any lines or angles are given equal, represent them as such in the figure, but when two other lines or angles in your figure seem to be equal be very careful not to assume that they really are so. Avoid drawing lines or angles in the figure equal which are not necessarily so. If a triangle is mentioned do not draw it as either isosceles or right angled. If an isosceles triangle, do not draw it equilateral. If a parallelogram, do not draw a rectangle. If a rectangle, do not draw a square. If a quadrilateral, let all its sides be unequal and none of its angles right angles. A pentagon is not a regular pentagon unless it is so stated. When two circles are given, let their radii be unequal unless the contrary is expressly stated. When three lines in your figure are concurrent, or three points lie on a straight line, assume this to be an accident until you have proved the contrary.

(2) **Note what theorems are suggested** by the hypothesis and by the required conclusion.

Thus if the hypothesis mentions or implies that one side of a given triangle is greater than another, it is almost certain that T.3 will be required in the proof; if it is required to prove that two lines are parallel, the result will probably be obtained by using P.1, P.2, or P.3; but the two lines may in reality be sides of a parallelogram, in which case the result might be proved by Pm.1*c*, or Pm.2.

(3) Try to find any facts about the figure which can be proved from the given hypothesis; and any possible facts about the figure *from which* the required conclusion could easily be proved.

In other words try to work on a few steps from the hypothesis, and to work *back* a few steps from the required conclusion.

(4) If the proof is not yet obvious, it requires more than the mechanical application of any method.

The student is necessarily thrown upon his own resources; but one further hint will frequently prove of the greatest value.

(5) Make sure that you are using **all the facts given** in the hypothesis. It is useless attempting to prove a rider without doing this.

Suppose, for example, that two angles are given equal. Any result which can be proved without using this fact would hold if the angles were not equal; but presumably the required result is *not* true if the angles are unequal.

It is needless to add that no student can expect to become expert at solving riders by any purely mechanical method. Considerable practice and considerable ingenuity are required before difficult riders can be attempted with success.

145.—The following examples are good illustrations of the method of solving riders. In these solutions the different numbered paragraphs correspond to the different numbered hints in §144.

Example 1.—*If two triangles ABC, DBC on opposite sides of the base BC are equal in area, the line AD is bisected by BC or BC produced.*

(1) *Cons.* — Draw the diagram as in Fig. 216. The lengths BA, AC, BD, DC are purposely made unequal. There are no lines or

angles to be marked as given equal; there is no convenient method of marking the equal areas, so that it is advisable to write down the hypothesis as well as the conclusion.

Given that $\triangle\ BAC = \triangle\ BDC$,
Required to prove that $AL = LD$.

(2) *Analysis.*†—*The hypothesis* (equal triangles on the same base) suggests Ar.5*c*.; but these triangles are *not* on the same side of this base. The *conclusion* ($AL = LD$) suggests the use of one of the congruence theorems, or P.5, or T.7; but in Fig. 216 none of these can be applied.

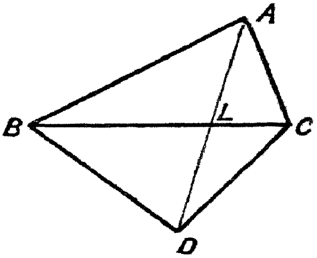

FIG. 216.

(3) Since $\triangle$s ABC, DBC are equal and on the same base, it follows by Ar.5*c*, Cor.2, that their altitudes are equal. Draw these altitudes AK, DM, each $\perp$ BC: (Fig. 217). Since $AK = DM$ it is easy to prove (by C.2) that $\triangle\ AKL \equiv \triangle\ DML$; whence $AL = DL$.

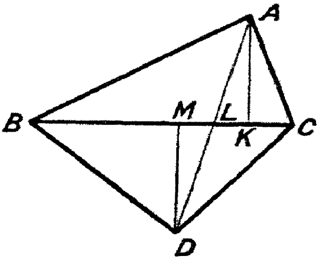

FIG. 217.

NOTE.—In writing out the solution of a rider, do *not* give the *Analysis*, but only the *Cons.* and *Proof* in the usual way.

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† The *analysis* of a geometrical theorem or problem is a discussion of the data and required conclusion or construction with a view to obtaining the *synthesis*, that is the logical arrangement of these data in argument so as to lead directly to the required conclusion or construction.

Example 2.—*$ABCD$ is a parallelogram, and P is any point within the triangle ABC. Prove that the triangles PAB, PAC are together equal in area to the triangle PAD.*

(1) *Cons.*—Draw the diagram as in Fig. 218. It is very likely that you will make BPD look like a straight line; if so, in order to *avoid* the mistake of *assuming in your argument* that BPD is straight, draw another figure.

Given that $ABCD$ is $\|^{\text{m}}$,
Required to prove that $\triangle PAB + \triangle PAC = \triangle PAD$.

(2) *Analysis.*—The *hypothesis* (a given parallelogram) suggests many theorems; but as the rider involves areas, we should expect to use Ar.2, Ar.3, or Ar.4.

The *conclusion* (two triangles equal to a third) does not suggest anything of value, except the use of theorems on area.

(3) It follows from the hypothesis that

$\triangle ABC = \triangle ADC = \frac{1}{2} \|^{\text{m}} ABCD$ (Pm.1).

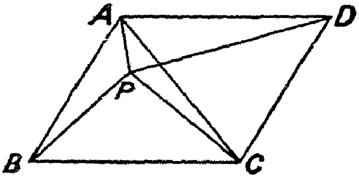

Fig. 218.

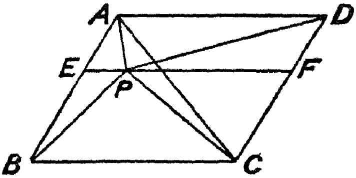

Fig. 219.

(4) No further hints can be obtained from Fig. 218; it is therefore advisable to introduce some new lines. Such new lines should be lightly drawn in pencil, and only inked in when it is certain that they are required. You will not succeed until you happen to guess the right construction.

(5) The fact that $ABCD$ is a parallelogram *must* be used. This may suggest a line EF through P parallel to AD (Fig. 219), which divides the given parallelogram into *two parallelograms*, viz. $AEFD$ and $EBCF$.

Now $\triangle PAD = \frac{1}{2} \|^{\text{m}} AEFD$, and $\triangle PBC = \frac{1}{2} \|^{\text{m}} EBCF$. Ar.3.
Hence $\triangle PAD + \triangle PBC = \frac{1}{2} \|^{\text{m}} ABCD = \triangle ABC$ Pm.1.
$= \triangle PAB + \triangle PAC + \triangle PBC$;
that is, $\triangle PAD + \triangle PBC = \triangle PAB + \triangle PAC + \triangle PBC$.
From these equals take the common triangle PBC;
thus $\triangle PAD = \triangle PAB + \triangle PAC$.

This proves the required result. In writing out the rider give only the construction and the portion in large print.

EXERCISES XLIX (Riders).

(Note that a parallelogram is often named by quoting the letters which mark two opposite angular points. Thus in Fig. 220 $\|^{m}$ *ABCD* may be called either $\|^{m}$ *AC* or $\|^{m}$ *BD.*)

1. Show that triangles on equal bases in the same straight line and with a common vertex are equal in area. Ar.5.

2. A parallelogram is divided into four equal areas by its diagonals. Pm.4 ; Ar.5.

3. *ABC* and *ABD* are two triangles on opposite sides of *AB.* Given that *AB,* or *AB* produced, bisects *CD,* prove that the two triangles are equal in area. Ar.5.

4. Triangles or parallelograms which are equal in area and between the same parallels have equal bases. Pm.1 ; Ar.5.

5. If a quadrilateral is divided by its diagonals into four equal triangles it is a parallelogram. Ques.4 ; Pm.4*c*.

6. *ABCD* is a trapezium, of which the side *AB* is parallel to the side *DC.* Show that the area of *ABCD* is equal to the area of a parallelogram formed by drawing through *M,* the middle point of *BC,* a straight line parallel to *AD.* P.1*c* ; C.2.

7. If two triangles have two sides of the one equal to two sides of the other, each to each, and the sum of the angles contained by these sides equal to two right angles, the triangles are equal in area. A.1*c* ; Ar.5.

8. Two straight lines *AB, CD* intersect in *E,* and the triangle *AEC* is equal to the triangle *DEB* ; show that *AD* and *CB* are parallel. Ar.5*c*.

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9. (In Fig. 220 *ABCD* is a parallelogram ; *EOF* and *GOH* are parallel to the sides. In this figure *AEOG* and *OHCF* are called "parallelograms about the diagonal" *AC; EBHO* and *GOFD* are called the "complements" of these parallelograms.)

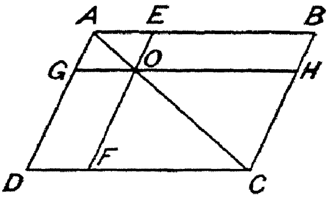

FIG. 220.

Prove that the complements of parallelograms about a diagonal of a parallelogram are equal in area. Pm.1.

(For instance in Fig. 220 prove *EBHO* = *GOFD.*)

10. In Fig. 208 if BG is drawn perpendicular to CD show that $AB \times BG = CB \times BE$, (*i. e.* the *product of the measures* of AB and BG is equal to the *product of the measures* of CB and BE).

11. Show that the three results obtained by multiplying the length of each side of a triangle by its perpendicular distance from the opposite vertex should be equal.

12. Prove that if two equilateral triangles are equal in area they are also congruent (Superposition).

13. [**Pm. 8.**] **Prove that, if two squares are equal in area, the straight lines on which the squares are described are also equal** (**Superposition**).

14. In Fig. 220 prove that the triangle GOC is equal to half the parallelogram EH.

15. In Fig. 220 prove that the triangle AEC is equal to the triangle AGC.

16. †Triangles on equal bases and of equal altitude are equal in area. Prove this (i) by means of Theorem Ar. 3, Corollary, or (ii) by means of Theorem Ar. 5.

17. †Parallelograms on equal bases and of equal altitude are equal in area. Prove this (i) by means of Theorem Ar. 2, Corollary, or (ii) by means of Theorem Ar. 4.

18. Parallelograms about the diagonal of a square (Ques. 9) are squares.

19 ABC is a triangle whose sides AB and AC are bisected at F and E; show that the triangle BFC is equal to the triangle BEC, and hence that FE is parallel to BC.

20. ABC is a triangle whose sides AB and AC are bisected at F and E; show that (1) the triangles BFE, AFE and CFE are all equal, and therefore (2) FE is parallel to BE.

21. ABC is a triangle, and D is the middle point of BC. G is any point on AD. Prove triangles AGB, AGC are equal.

22. Show how to find a point G in a triangle ABC such that the three triangles AGB, BGC, CGA are equal.

23. If O is the middle point of AC (Fig. 220) prove that the complements are congruent.

24.* $ABCD$ is a given quadrilateral. Find the locus of a point P which moves so that the quadrilateral $ABCP$ is always equal in area to $ABCD$.

25.* ABC, ADE are two intersecting straight lines and F is a point within the angle CAE. Find the locus of a point P which moves so that $\triangle PBC + \triangle PDE = \triangle FBC + \triangle FDE$.

---

† Logically the second method is preferable, as it does not involve the question of arithmetical measurement and incommensurables.

26. On the sides *AB, AC* of a triangle, parallelograms *ABDE, ACFG* are constructed outside the triangle ; if *DE, FG* cut in *H*, prove that the two parallelograms are together equal to the parallelogram having, as adjacent sides, *BC* and a straight line equal and parallel to *AH*.

27.* If *P* be any point without the parallelogram *ABCD*, show that the sum or difference of the triangles *PAB, PCD* is half the parallelogram *ABCD*.

28.* *ABCD* is a parallelogram and *P* is a point lying within the angle formed by *DA* produced and *AB*. Prove that the sum of the triangles *PAB, PAD* is equal to the triangle *PAC*.

29.* *ABCD* is a parallelogram and *P* is a point lying without the parallelogram but within the angle *BAD*. Prove that the difference of the triangles *PAB, PAD* is equal to the triangle *PAC*.

146. Determination of areas by diagrams drawn to scale.

—The following examples will show the different methods which can be used for determining areas by diagrams drawn to scale.

Example 1.—*Determine the area of a triangular plot of ground whose sides measure* 16 *metres,* 20 *metres,* 24 *metres respectively.*

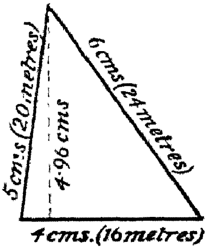

Fig. 221.

A convenient scale of representation is 4 metres to 1 cm. The three sides of the triangle will thus be represented by lengths 4, 5, 6 cms. respectively. Draw this triangle (Fig. 221).

Taking any side (say the 4 cms.) as base, draw and measure the altitude.

The result is 4·96 cms. But 1 cm. represents 4 metres, hence altitude of triangular plot is $4 \cdot 96 \times 4$ metres.

Thus area of triangular plot $= \frac{1}{2}$ base × altitude. Ar.3.

$= \frac{1}{2} \times 16 \times (4 \cdot 96 \times 4)$ square metres

$= 15{,}900$ square metres (correct to three significant figures).

Note 1.—In this method we determine any required length on the given figure by measuring the corresponding length on the diagram drawn to scale.

Note 2.—No result which is obtained by a process involving measurement or counting of squares can be regarded as correct to more than three figures. Accordingly not more than three significant figures should be given in the answer.

Example 2.—*Find the area of a circle of diameter* 40 *miles.*

A convenient scale of representation is 20 miles to the inch. Draw on tenth-inch squared paper a circle of diameter 2 inches (see Fig. 196).

Counting as in § 131 we find that the area of the circle is equivalent to 312 squares.

Now each side of a square measures $\frac{1}{10}$ inch and therefore represents 2 miles. Thus the area represented by each square is 2 × 2 square miles (Ar. 1).

Hence total area of circle $= 312 \times (2 \times 2)$ square miles.
$= 1250$ sq. miles (to three sig. figures).

NOTE.—In this method we count the number of squares in the diagram drawn to scale and determine *what area each square represents.*

EXERCISES L.

BY CONSTRUCTION AND CALCULATION.

1. Find the area of a triangular board whose sides measure 20″, 30″, and 40″.

2. A man owns a farm in the shape of a parallelogram in which two adjacent sides measure 2·2 and 3·6 miles respectively and contain an angle of 70°. How many square miles are there in the farm?

3. A circular lake measures 30 yards in diameter. Find its area.

4. A grass lawn is in the shape of an isosceles triangle, on a base 10 yards, with a vertical angle of 120°. Find its area.

5. Find the area of a triangle whose sides measure 4 ft., 4 ft., 5 ft.

6. Find the area of the portion of a circle of radius 10 ft. included between two radii which form an angle of 60°.

7. Find the area of an equilateral triangle whose sides each measure 0·01 inch. [Use a scale of 0·005 inch to 1″.

8. Find the area of a quadrant of a circle of radius 0·1 inch. [Use a scale of 0·1 inch to 1″.

9. A copy of a rectangle is drawn to a scale of $\frac{1}{4}$. What fraction is each side of the copy of the corresponding side of the given rectangle? What fraction is the area of the copy of the area of the given rectangle?

10. A copy of a triangle is drawn to a scale of $\frac{1}{3}$. What fraction is the base of the copy of the base of the given triangle? What fraction is the altitude of the copy of the altitude of the given triangle? What fraction is the area of the copy of the area of the given triangle?

CHAPTER VIII.

PROBLEMS ON AREA.

147. PROBLEM.—**To divide a parallelogram or a triangle into any given number of equal parts.**

(i) *Required* to divide $\|^{\text{m}}$ *ABCD* into n equal parts.

Cons.—Divide the base *DC* (Fig. 222) into n equal parts, *DE, EF*, . . . § 114.

Through *E, F*, . . . draw lines *EK, FL*, . . . $\parallel$ *DA* and terminated by *AB*.

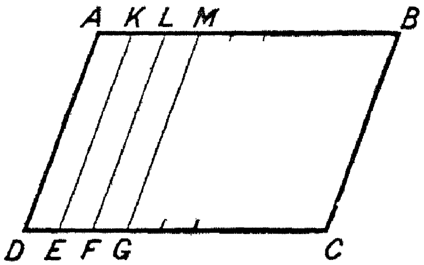

FIG. 222.

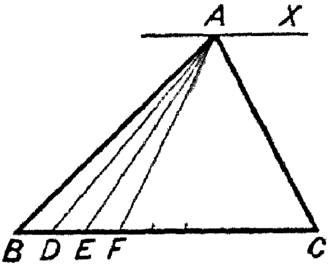

FIG. 223.

Proof.—In $\|^{\text{m}}$ *ABCD, CD* $\parallel$ *BA*. Also, *EK, FL*, . . . are $\parallel$ *DA*. *Cons.*

Hence *CB, EK, FL*, . . . are $\parallel$ each other. P.3.

Also *AB* $\parallel$ *DC*; hence each of the figures *AE, KF, LG*, . . . is $\|^{\text{m}}$.

These $\|^{\text{ms}}$ are on equal bases *DE, EF*, . . . and between the same parallels *DC, AB*.

Hence $\|^{\text{ms}}$ *AE, KF, LG*, . . . are equal in area. Ar.4.

(ii) *Required* to divide $\triangle$ *ABC* into n equal parts.

Cons.—Divide the base *BC* (Fig. 223) into n equal parts, *BD, DE*, . . . § 115.

Join the vertex *A* to each of the points *D, E*, . . .

Proof.—Through A draw $AX \parallel BC$.
$\because$ $\triangle$s ABD, ADE, . . . are on equal bases BD, DE, . . . and between the same $\parallel^s$ BC, AX;
hence $\triangle$s ABD, ADE, . . . are equal in area. Ar. 5.

148. PROBLEM.—**To construct a triangle equal to a given convex polygon.**

Required to construct a triangle equal to the given polygon $ABCD$ . . . (Fig. 224).

(i) To obtain an equal polygon with the number of sides diminished by one.

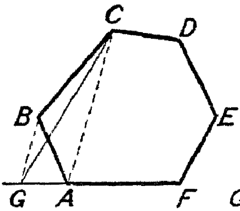

FIG. 224.

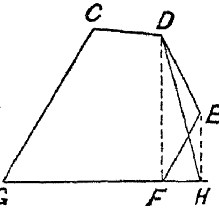

FIG. 225.

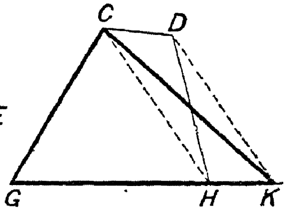

FIG. 226.

Cons.—Join CA. Draw $BG \parallel CA$ meeting FA produced in G. Join CG. § 111.

Then the polygon GCD . . . shall be equal to the given polygon; and it has fewer sides by one.

Proof.—$\because$ $\triangle$s AGC, ABC are on the same base CA and between the same parallels CA, BG,
$\therefore$ $\triangle AGC = \triangle ABC$. Ar. 5.
To each of these equal areas add the figure ACD . . .
hence polygon GCD . . . = polygon $ABCD$ . . .

(ii) By applying the same construction to the polygon GCD . . . another polygon can be obtained of equal area and having fewer sides by one.

This process can be repeated till the number of sides of the polygon is reduced to three; *i. e.* till a triangle is obtained equal in area to the original polygon.

The various stages in the process are indicated in Figs. 224, 225, 226. Thus in Fig. 224 $ABCDEF$ gives the equal polygon $GCDEF$; in Fig. 225 $GCDEF$ gives the equal quadrilateral $GCDH$; and in Fig. 226 $GCDH$ gives the equal $\triangle GKC$. In practice the various steps of the construction should be done on one figure.

NOTE.—In this case the polygon $ABCDEF$ has been reduced to a triangle whose vertex lies at C and whose base lies along AF. A little reflection will show the student that by means of this same method the polygon can be reduced to a triangle having its vertex at the intersection of any two adjacent sides of the polygon and its base along any of the remaining sides.

EXERCISES LI.

1. Draw $\triangle ABC$ making $AB=1{\cdot}5''$, $BC=2''$, $CA=2{\cdot}5''$. Divide the triangle into three equal triangles by lines drawn through A. Measure the sides of the middle triangle.

2 Draw $\triangle HKL$ making $HK=5$ cms., $KL=9$ cms., $LH=7{\cdot}2$ cms. Divide the triangle into three equal parts by two lines drawn from the point H. Measure these two lines.

3. $ABCD$ is a convex quadrilateral in which $AB=1{\cdot}4''$, $C=2{\cdot}6''$, $BCD=2{\cdot}8''$, $DA=1{\cdot}8''$, $DB=2{\cdot}4''$. Find the area of this quadrilateral (i) by squared paper; (ii) by adding the areas of the triangles ABC, CDA; (iii) by adding the areas of the triangles ABD, CBD; (iv) by reducing it to an equal triangle and calculating the area of the latter.

4. $PQRS$ is a convex quadrilateral in which $PQ=3{\cdot}3''$, $QR=2{\cdot}6''$, $RS=1\ 8''$, $SP=3''$, $\angle S=120^\circ$. Find the area of this quadrilateral i) by squared paper; (ii) by adding the areas PQR, PSR; (iii) by adding the areas PQS, RQS; (iv) by reducing the quadrilateral to an equal triangle and calculating the area of the latter.

5. $ABCDE$ is a convex pentagon in which $AB=3$ cms., $BC=3{\cdot}2$ cms., $CD=4{\cdot}6$ cms., $DE=3$ cms., $EA=3{\cdot}2$ cms., $AC=4{\cdot}4$ cms., $AD=4{\cdot}8$ cms. Reduce the figure to an equal triangle having its vertex at A and its base along CD. Measure the base of this triangle.

6. Construct a pentagon $ABCDE$ in which each side measures 3 cms. and each angle 108°. Reduce the pentagon to an equal triangle ABF having AB for one side and $\angle B$ for one angle. Measure BF.

7. Construct a hexagon $ABCDEF$ in which each side measures $1''$ and each angle 120°. Reduce the hexagon to an equal triangle APQ in which the vertex is at the point A and the base PQ lies along the line CD. Measure PQ.

8. Draw a circle of radius $5{\cdot}3$ cms. and in it inscribe a heptagon $ABCDEFG$ each of whose sides measures $4{\cdot}6$ cms. Reduce the heptagon to an equal triangle and hence determine its area

149. *Prel. Ex.*—Given that BCE, BFD are two straight lines drawn from the point B, and that DE is parallel to FC; prove that the triangles BDC, BFE are equal.

PROBLEM.—**To construct a triangle equal to a given triangle, given one side and an adjacent angle of the required triangle.**

Required to construct a triangle, equal in area to *ABC* (Fig. 227), having one angle equal to *X*, and an adjacent side equal to *Y*.

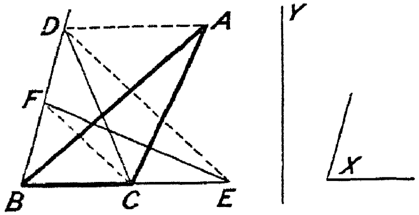

FIG. 227.

(i) Construct a triangle equal to the given triangle, and having an angle equal to ∠ *X*.

Cons.—At *B* in *BC* make ∠ *CBF* = ∠ *X*. § 109.
Through *A* draw *AD* ∥ *BC* meeting *BF* at *D*. § 111.
Then *BDC* is the required triangle.

Proof.—∵ △s *DBC, ABC* are on same base *BC* and between same parallels *BC, AD*; therefore △ *DBC* = △ *ABC*. Ar. 5.
Also ∠ *DBC* = ∠ *X*.

(ii) Hence construct a triangle equal to the given triangle, having an angle equal to *X*, and an adjacent side equal to *Y*.

Cons.—From *BC*, or *BC* produced, cut off *BE* equal to *Y*. Join *DE*.
Through *C* draw *CF* ∥ *ED*, meeting *BD* (produced if necessary) at *F*. Join *EF*. § 110.
Then *FBE* is the required triangle.

Proof.—∵ △s *EFC, DFC* lie on the same base *FC* and between the same parallels *FC, DE*,
∴ △ *EFC* = △ *DFC*. Ar. 5.
To each add △ *BFC*, ∴ △ *FBE* = △ *DBC*. **Hence** △ *FBE* = △ *ABC*.
Also ∠ *FBE* = ∠ *X*, and side *BE* = *Y*.

COROLLARY 1.—**To construct a parallelogram equal to a given triangle given one angle and an adjacent side of the parallelogram.**

First bisect the given triangle.

Construct a triangle PQR (Fig. 228) equal to the half of the given triangle, and having $\angle Q$ and adjacent side QR respectively equal to the given angle and line.

Draw $PS \parallel QR$, and $RS \parallel QP$.

Then $PQRS$ is the required $\parallel^{m}$; for it is double of $\triangle PQR$ and therefore equal to the given triangle.

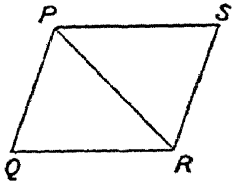

FIG. 228.

COROLLARY 2.—**To construct a triangle (or parallelogram) equal to a given polygon, given an angle and an adjacent side of the required triangle (or parallelogram).**

(i) Construct a triangle equal to the given polygon.

(ii) Construct a triangle (or parallelogram) equal to the triangle in (i) having the given angle and adjacent side.

EXERCISES LII.

PRACTICAL GEOMETRY.

1. The base of a triangle measures 3″ and its two base angles measure 30° each. Construct an equal triangle on the same base having one base angle of 90°. Measure the other base angle.

2. The sides of a triangle measure 3, 4, 5 cms. Construct an equal triangle in which two sides measure 4 cms. Measure the angle included between these sides.

3. Construct a triangle HKL having $HK = 1{\cdot}5''$, $KL = 2''$, $LH = 2{\cdot}5''$. Construct an equal triangle having one angle equal to L and an adjacent side 3″. Measure the other adjacent side.

4. Construct a triangle ABC, having $BC = 2{\cdot}6''$, $\angle B = 60°$, and equal in area to an equilateral triangle of side 2″. Measure AB.

5. Construct a triangle DEF, having $EF = 5$ cms., $\angle E = 37°$, and equal in area to a square of side 3 cms. Measure DE.

6. Construct a parallelogram having two sides, 1·5″ and 2·5″, containing an angle 72°. Construct an equal parallelogram having a side 2″ and an angle 54°. Measure the other side.

7. Construct a rhombus of side 1·6″, having an angle 110°. Construct an equal rectangle having one side 1″. Measure the other side.

8. Construct a quadrilateral *ABCD* having *AB*=1·8″, *BC*=2·6″, *CD*=1″, *DA*=3″, *BD*=2·4″. Construct an equal rectangle having one side 1·4″. Measure the other side.

BY CONSTRUCTION AND CALCULATION.

9. Construct an isosceles triangle having a base = 3·6″ and an area = 4·32 sq. inches. Measure the sides.

10. Construct a parallelogram having a base = 2·2″, an angle = 70°, and an area = 3·3 sq. inches. Measure the side.

11. Construct a triangle having an area of 15 sq. cms., an angle = 50°, and one side adjacent to this angle = 6 cms. Measure the side opposite to this angle.

12. Construct a triangle in which two sides measure 4 and 4·5 cms. respectively, and the area is 7 sq. cms. Show that from these data two different triangles can be drawn, and measure the third side in each case.

150. **On the Solution of Problems.**—The hints on the Solution of Riders which were given in the preceding chapter (§ 144) apply whether the rider is a theorem or a problem. If, however, the rider is a problem one of the following two special methods can usually be applied.

FIRST METHOD.—*Intersection of Loci.*

When the solution of the problem consists of, or depends upon, finding the position of a certain point, this point can often be found as the intersection of two loci.

This method has already been explained and illustrated in § 124.

SECOND METHOD.—*Inference from the required result.*

In this method a figure is drawn representing the problem as solved. Certain facts are then deduced from this figure (usually with other lines added to it) which suggest the required construction.

The student is recommended to draw the original figure (representing the problem solved) in *ink*, and any lines he may add (for trial) in *pencil* so that they may be readily removed if wrong.

Example.—*Bisect the area of a triangle by a straight line drawn through a given point in the base of the triangle.*

Cons.—(1)† In the base BC of $\triangle ABC$ (Fig. 229) take a point P. Do *not* take P near the middle of the base; there will then be no fear of *assuming* inadvertently that P bisects the base.

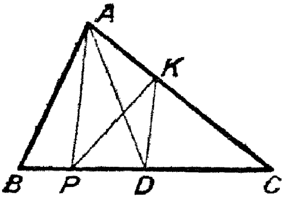

FIG. 229.

Suppose that PK is the line which bisects the area. It is required to find some fact about the figure which will enable us to draw PK correctly. We are assuming that

$$\triangle PKC = \text{quad. } ABPK = \tfrac{1}{2} \triangle ABC.$$

(2) At present we have only the triangle ABC and the line PK. The assumption that PK bisects the area of the triangle does not suggest the use of any theorem.

(4) Obviously some new lines must be added to the figure. Various constructions must be tried till some fact of value is obtained.

(5) Remembering that we must use the assumption that PK bisects the area of the triangle, it is natural to bisect the area by some new line, and then see if we can obtain any result by comparing the halves formed by PK with the halves formed by the new line. Now if D bisects BC, then *median AD bisects the area ABC* (Ar. 5, Cor. 2).

Thus $\triangle ABD = \triangle ADC = \tfrac{1}{2} \triangle ABC.$

Hence $\triangle ADC = \triangle PKC.$

From these equal triangles subtract the common triangle DKC.
Thus $\triangle ADK = \triangle PDK$; whence $AP \parallel DK$. Ar. 5c.
But the points A, P, D are all known. Hence *the point K can be determined by drawing $DK \parallel PA$.*

Thus we obtain the following solution:—

Cons.—Join AP. Bisect the side BC at D.
Draw $DK \parallel PA$, meeting AC (or AB) at K. Join AD, PK.
Then PK bisects the area of the triangle.

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† These numbers refer to the numbered hints in § 144.

Proof.—∵ △s *PDK, ADK* are on the same base *DK* and between the same parallels *DK, PA,*

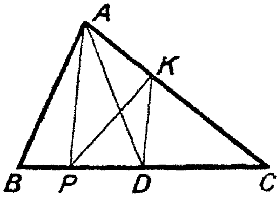

FIG. 229.

∴ △ *PDK* = △ *ADK.* Ar.5.

To each add △ *DKC*; thus

△ *PKC* = △ *ADC.*

But △ *ADC* = △ *ADB* (Ar.5)

$= \frac{1}{2}$ △ *ABC.*

Hence △ *PKC* = $\frac{1}{2}$ △ *ABC.*

NOTE.—The student should also try to solve the problem using (i) the median through *B,* and (ii) the median through *C,* instead of the median through *A*—keeping *P* near to *B* in each case. He should moreover find why the construction given above is to be preferred.

EXERCISES LIII (Riders).

1. *ABC* is a given triangle; show how to construct a right-angled triangle equal in area to *ABC.*

2. *ABC* is a given triangle, and *DE* a given straight line which is not parallel to *AB*; show how to construct on *AB* a triangle equal in area to *ABC* and having its vertex in *DE.*

3. *ABC* is a given triangle; show how to construct a triangle whose area is double the area of *ABC.*

4. Construct a rectangle equal to a given parallelogram and having one side equal to a side of the parallelogram.

5. On one side of a given triangle construct an isosceles triangle equal to the given triangle.

6. Construct a triangle equal to a given parallelogram, having an angle equal to a given angle.

---

7. Prove the following construction for drawing a parallelogram equal to a given triangle *ABC* and having an angle equal to a given angle *X*:—

Bisect *BC* at *D.* Draw *AZ* ∥ *BC.*

Draw *DE* making ∠*EDC* = ∠*X*; draw *CF* parallel to *DE.* If *DE, CF* meet *AZ* in *E, F* then *DEFC* is the required parallelogram. Ar.3; Ar.5.

8. Show how to divide a parallelogram into three equal parts by lines drawn through one angular point. Pm.1.

9. Show how to bisect the area of a given parallelogram by a line drawn through a given point in one of its sides. § 112; C.2; Pm.1.

10. *ABC* and *ADE* are two perpendicular straight lines; also *BD* is parallel to *CE.* Prove that △ *ABE* = △ *ACD,* and hence that *AB* × *AE* = *AC* × *AD.* Ar.3; Ar.5.

11. Prove the following construction for bisecting the area of a convex quadrilateral *ABCD* by a line drawn from *A*:—

Bisect *BD* at *E*. Draw *EF* *AC* meeting *BC* or *CD* at *F*. Join *AF*. Ar.5.

12. *ABCDEF* is a polygon having a re-entrant angle at *B*. Prove that the construction given in Part (i) of the problem in § 148 will usually apply to this figure, and mention the case when it fails. Ar.5.

13. *ABCD* is a four-sided figure such that *AB* and *DC*, the sides, are equal, and *AD* and *BC*, the top and bottom are parallel. Show how to construct on base *BC* an isosceles triangle equal to *ABCD*.

14. On a given line construct a rectangle equal to a given rectangle.

15. Construct a rectangle equal to the sum of two given rectangles.

16. Construct a rectangle equal to the difference of two given rectangles.

17. Show how to bisect the area of a parallelogram by a straight line (1) drawn through a given point, *or* (2) drawn parallel to a given line, *or* (3) touching a given circle.

18. Trisect a triangle by lines drawn through a point in one of the sides.

19. Through a point in the side of a triangle draw a line cutting off one-fourth, one-fifth, or any required submultiple of the area.

20. Construct a triangle equal to a given triangle and having two of its sides of given length. When is this impossible?

21. Construct a parallelogram equal to a given parallelogram and having two of its sides of equal length.

22. State and prove the converse of the theorem in Ex. 1, § 145. Hence show how to find a point *P* within a triangle *ABC* such that the triangles *PBC, PCA, PAB* are equal in area.

23. Find a point *P* within a quadrilateral *ABCD* such that the triangles *PAB, PBC, PCD* are all equal in area.

24.* Trisect a parallelogram by lines drawn through a point in one of the sides.†

25.* Bisect a quadrilateral by a line drawn through a point in one of the sides.†

26.* Through an angular point of a quadrilateral draw a line cutting off one-third, one-fourth, or any required submultiple of the area.

27.* Through a point in the side of a quadrilateral draw a line cutting off one-third, one-fourth or any required submultiple of the area.†

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† In these problems it is difficult to avoid a large number of different cases. Perhaps the simplest methods can be obtained by first reducing the figure to an equal triangle having its vertex at the given point and its base along the opposite side.

151. **Multiplication and Division.**—Simple cases of Multiplication and Division can be performed by Geometrical construction.

PROBLEM.—**To find geometrically the value of $\frac{a \times b}{c}$.**

Cons.—Draw two ⊥ lines OX, OY (Fig. 230).
From OX mark off $OA = a$ units, and $OC = c$ units. From OY mark off $OB = b$ units. Join CB.

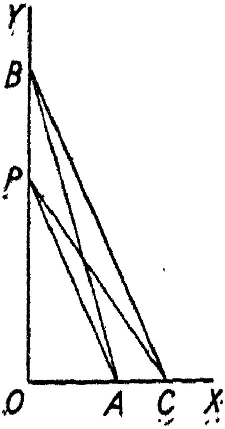

FIG. 230.

Draw $AP \parallel CB$. Then $OP = \frac{a \times b}{c}$.

Proof.—$\triangle PCA = \triangle PBA$. Ar. 5.
To each add $\triangle OPA$; thus $\triangle OPC = \triangle OAB$.

Hence $\frac{1}{2} OP \times OC = \frac{1}{2} OA \times OB$, Ar. 3.
or $\frac{1}{2} OP \times c = \frac{1}{2} a \times b$.

Whence $OP = \frac{a \times b}{c}$.

NOTE.—Another method of viewing this problem will be indicated in Part II, under the head of *Proportion*. It will there be shown that the lines OX, OY need not be at right angles.

Example.—*Evaluate* (i) $2 \cdot 4 \times 1 \cdot 3$, (ii) $2 \cdot 4 \div 1 \cdot 3$.

Use the method of the preceding Problem :—
(i) making $a = 2 \cdot 4$, $b = 1 \cdot 3$, $c = 1$. *Result*, 3·12.
(ii) making $a = 2 \cdot 4$, $b = 1$, $c = 1 \cdot 3$. *Result*, 1·85.

EXERCISES LIV.

Evaluate geometrically :—

1. $17 \times 29 \div 19$.
2. $1 \cdot 7 \div 2 \cdot 9 \times 1 \cdot 9$.
3. $2 \cdot 1 \times 3 \cdot 2$.
4. $2 \cdot 3 \div 1 \cdot 3$.
5. $1 \cdot 5 \times 4 \cdot 1 \div 3 \cdot 2$.
6. $2 \cdot 3 \times 3 \cdot 4$.
7. $4 \cdot 5 \div 1 \cdot 7$.
8. $7 \cdot 3 \div 4 \cdot 2$.

CHAPTER IX.

THE RIGHT-ANGLED TRIANGLE

152 **The Squares on the Sides of a Triangle**—This chapter deals with the areas of the three squares which can be described on the three sides of any triangle (Fig 231) It will be shown that there is a *very simple* (and very important) relation between these areas when the triangle is right-angled.

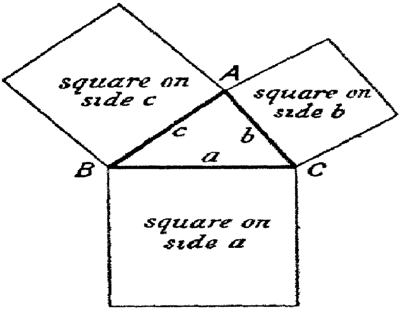

Fig 231

To find the area of any rectangle we multiply the lengths of two adjacent sides [Ar 1] Hence to find the area of a square we multiply the length of a side *by itself*; for the sides of a square are all equal.

In this chapter we shall frequently use the letters *A, B, C* to denote the three angles of a triangle, and the letters *a, b, c* to denote the lengths of the three sides; the side whose length is *a* being *opposite* to the angle *A, b* opposite to *B,* and *c* opposite to *C.* See Fig 231

EXERCISES LV.

BY DRAWING AND CALCULATION.

1. Draw three triangles in each of which $b=1{\cdot}2''$, $c=1{\cdot}6''$ (see (Fig. 231). Make ∠*A* obtuse in the first triangle, acute in the second, and *an exact right angle* in the third.

In each case determine accurately the areas of the three squares. (Note that the areas can be determined without *drawing* the squares.)

In each case determine whether the sum of the squares on b and c is greater than, equal to, or less than the square on a.

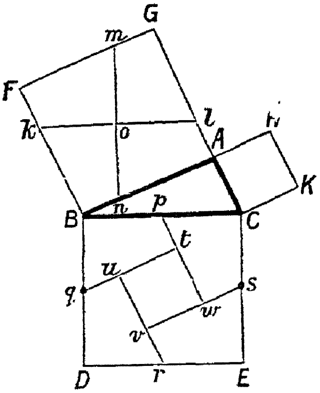

FIG. 232.

2. What simple relation appears to hold between the squares on the sides of a right-angled triangle?

3. Construct a triangle making the angle *A* an exact right angle, and making the sides *b* and *c* roughly about 1″ and 1·5″. Draw the three squares accurately.

Draw through *A* a line accurately perpendicular to *a*, and produce this line so as to divide the square on *a* into two rectangles.

Determine the area of each of the squares on *b* and *c* and of each portion of the square on *a*. *What do you notice about your results?*

4. Construct an *acute-angled* triangle, making the sides about 4, 4·5, and 5 cms. Draw accurately the squares on the three sides.

Divide each square into two portions by a line drawn through the opposite corner of the triangle accurately ⊥ the side of the square.

Determine the area of each portion of each square. *What do you notice about your results?*

5. Copy Fig. 232 making *BA* about 1·8″, *AC* about ·9″, and ∠*BAC* an exact right angle; draw the squares on the sides accurately.

Through the middle points p, q, r, s of the sides of the square BE, draw the lines ptw, qvt, rvu, swv parallel to the lines AC and AB alternately. Through O the middle point of BG, draw $kol \parallel BC$, and $mon \perp BC$.

Show by tracing paper, or otherwise, that the four parts of the square FA and the whole square AK can be fitted together to make up the square BE.†

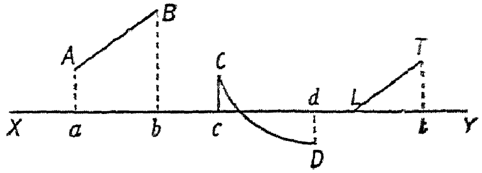

FIG. 233.

153. **On Projections.**—The projection of a given point on a given straight line is the foot of the perpendicular drawn from the point to the line. Thus in Fig. 233 the projection of the point B on the line XY is the point b.

The projection of a line on a given straight line is that portion of the given straight line which lies between the projections of the two extremities of the given line. Thus in Fig. 233 the projection of the line AB on the straight line XY is the line ab.

The first of the two lines (the line which is projected) may be either *straight*, or *curved*. The second of the two lines (on to which the first is projected) *must however be a straight line*, since the definition involves perpendiculars to this line. Thus in Fig. 233 the projection of the curved line CD on the straight line XY is cd. Note also that the projection of the point L on XY is the point L itself; and the projection of the line LT is the line Lt.

---

† Many other devices have been invented for dissecting the squares on BA and AC into parts which fit together into the square on BC. The one given in the text is perhaps the easiest to remember though not the easiest to prove logically. In the set of Geometrical Models to which reference was made on p. 190, a dissection of the proposition is given in wooden models.

154. *Prel. Ex.*—Fig. 234 shows a triangle right-angled at A, with a square described on each side. Also AL is perpendicular to BC. Prove (i) $\triangle FBC \equiv \triangle ABD$ [C.1], and hence (ii) sq. $FA =$ rect. BL [Ar.3.]

THEOREM [RT.2].†—**In a right-angled triangle, the square described on the hypotenuse is equal in area to the sum of the squares described on the other two sides.**

Fig. 234 shows a triangle ABC, with a square described on each side:—

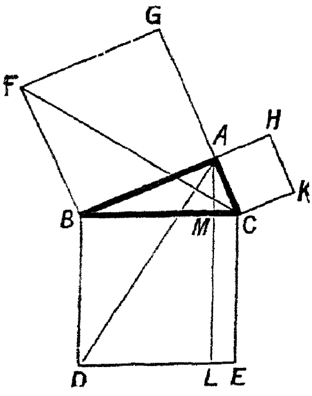

FIG. 234.

Given that $\angle BAC = \mathrm{R}$,
Required to prove that
sq. $BE =$ sq. $FA +$ sq. AK.

Cons.—Draw $AL \parallel BD$ (or CE). Join FC, AD.

Proof.—$\because$ adj. angles BAG, BAC are both right angles,
$\therefore$ GAC is a straight line. A.1c.
$\because$ sq. FA and $\triangle FBC$ are on the same base FB, and between the same parallels FB, GC,
$\therefore$ sq. $FA = 2 \triangle FBC$. Ar.3.
Similarly rect. $BL = 2 \triangle ABD$. Ar.3.

Again, $\angle FBA = \angle CBD$; to each add $\angle ABC$;
$\therefore$ $\angle FBC = \angle ABD$.

In $\triangle$s FBC, ABD,

$$\because \left\{\begin{array}{c} FB = AB, \\ BC = BD, \\ \angle FBC = \angle ABD, \end{array}\right\} \therefore \triangle FBC \equiv \triangle ABD. \qquad \text{C.1.}$$

But sq. $FA = 2 \triangle FBC$, and rect. $BL = 2 \triangle ABD$.
Hence sq. $FA =$ rect. BL.

Similarly it can be proved that
sq. $AK = 2 \triangle KCB$; rect. $CL = 2 \triangle ACE$;
and hence that sq. $AK =$ rect. CL.

But rects. BL, CL together make up sq. BE;
hence sq. $FA +$ sq. $AK =$ sq. BE.

---

† For Theorem RT.1, see p. 106.

COROLLARY [RT.3.]—**In a right-angled triangle the square on either side containing the right angle is equal to the rectangle contained by the hypotenuse and the projection of that side on the hypotenuse.**

For it has been proved that sq. on BA = rect. BL.
Also rect. BL = rect. (BD, BM) = rect. (BC, BM).

NOTE.—This particular proof of the theorem was discovered by Euclid. The theorem itself was first proved by Pythagoras, a celebrated Greek philosopher, B.C. 582–500. It is probable that the Egyptians knew the facts (apart from any proof) long before this date.

155. THEOREM [RT.2c].—**If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, the angle contained by these two sides is a right angle.**

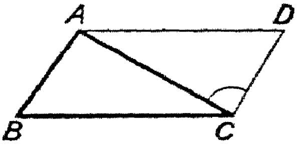

FIG. 235.

In Fig. 235:—*Given* that
sq. on BC = sq. on BA + sq. on AC,
Required to prove that $\angle BAC = R$.

Cons.—Draw $CD \perp CA$, making $CD = AB$. Join AD.

Proof.—$\because$ $CD = AB$, $\therefore$ sq. on CD = sq. on AB. Pm.7.
hence sq. on CD + sq. on AC = sq. on AB + sq. on AC.

In $\triangle ACD$, $\because$ $\angle ACD = R$,
$\therefore$ sq. on CD + sq. on AC = sq. on AD. RT.2.
Also sq. on AB + sq. on AC = sq. on BC. *Hyp.*
$\therefore$ sq. on AD = sq. on BC; whence $AD = BC$. Pm.8.

$$\text{in } \triangle\text{s } ABC, CDA, \left\{\begin{array}{l} AB = CD, \\ BC = DA, \\ AC = CA, \end{array}\right\} \therefore \angle BAC = \angle ACD. \quad \text{C.3.}$$

But $\angle ACD = R$ (*Cons.*); **hence** $\angle BAC = R$.

156. **Calculations.**—If the side of a square is a units of length, then the area of the square is $a \times a$, i.e. a^2, in the corresponding units of area. Thus Theorem RT.2 may be expressed algebraically by the formula

$$b^2 + c^2 = a^2,$$

where a is the hypotenuse. Given any two sides of a right-angled triangle we can now calculate the third.

Example 1.—*Find the hypotenuse of a right-angled triangle, if the other two sides measure* 5 *ft. and* 8 *ft. respectively.*

We have $b=5$, $c=8$, $a=?$

Hence from the formula $25+64=a^2$, i. e. $a^2=89$,
whence $a=\sqrt{89}$ or **9·43** ft.

Example 2.—*A ladder* 20 *ft. long rests against a window-sill at a height of* 17 *ft. from the ground. How far is the foot of the ladder from the wall?*

In Fig. 236 $BC=20$ ft., $AC=17$ ft. Thus $a=20$, $b=17$, $c=?$

Hence from the formula, $17^2+c^2=20^2$

$$c^2=20^2-17^2=(20+17)\,(20-17)=37\times3=111$$
$$c=\sqrt{111} \text{ or } \mathbf{10{\cdot}53} \text{ ft.}$$

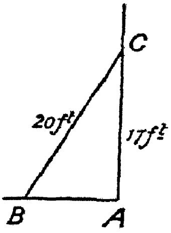

FIG. 236.

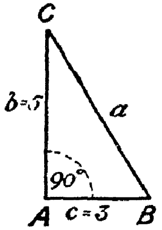

FIG. 237.

157. Square Roots by Geometrical Construction.—The following examples illustrate a geometrical method of obtaining the square root of an integer which is not an exact square. The first step is arithmetical, viz. to express the given integer as the sum or difference of two squares. The method is based on Theorem RT.2.

Example 1.—*Find the value of* $\sqrt{34}$.

We find (by guessing) that we can express 34 as the sum of two squares :—$34=5^2+3^2$.

Construct a triangle ABC (Fig. 237) in which $A=90°$, $b=5$ units of length, $c=3$ units. By measuring, find $a=5{\cdot}83$ units. Then $\sqrt{34}=\mathbf{5{\cdot}83}$.

Proof.— $a^2=b^2+c^2=5^2+3^2=34$. **Hence** $a=\sqrt{34}$.

NOTE.—Any convenient unit of length may be used. If squared paper is not used or if millimetre squared paper is used, the centimetre is the most convenient unit. If tenth-inch squared paper is used the inch (or the half-inch) will make a convenient unit.

Example 2.—*Find the value of* $\sqrt{7}$.

We find (by guessing) that we can express 7 as the difference of two squares :—$7=4^2-3^2$.

Take a right angle XAY (Fig. 238). From AY mark off AC of length 3 units. With centre C and radius 4 units of length, describe an arc cutting AX at B. By measuring we find $AB=2·65$. Then $\sqrt{7}=$ **2·65.**

Proof.— $c^2=a^2-b^2=4^2-3^2=7$. **Hence** $c=\sqrt{7}$.

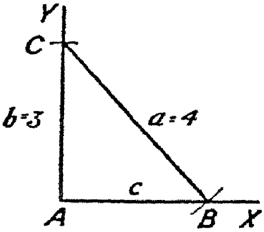

FIG. 238.

NOTE 1.—Obviously squared paper may be used with advantage.

NOTE 2.—If the side a is only a little longer than b a good result will not be obtained, as the arc and AX will not be sufficiently inclined to give a good intersection at B.

158. **Difference of two squares.**—It will be found that any number can be expressed as the difference of two squares, by factorising the number and using the formula $a^2-b^2=(a+b)(a-b)$.

To illustrate this take the numbers 22 and 43.

(1) Assume $22=a^2-b^2$, *i. e.* $11\times2=(a+b)(a-b)$.
Then assume $a+b=11$, $a-b=2$.
Solving these equations we obtain $a=6·5$, $b=4·5$.
Thus $22=6·5^2-4·5^2$.

(2) Assume $43=a^2-b^2$, *i. e.* $43\times1=(a+b)(a-b)$.
Then assume $a+b=43$, $a-b=1$.
Solving these equations we obtain $a=22$, $b=21$.
Thus $41=22^2-21^2$.

Applying the methods of § 157 we can now find $\sqrt{22}$ and $\sqrt{43}$.

In the second case, however, we shall find that we do not obtain a very accurate result for the reason mentioned in Note 2, § 157. This difficulty will occur whenever one of the factors of the given number is *very small* compared to the other; but it can usually be evaded by some modification of the method. It will often be found useful to use 5 as one factor. Thus if we factorise 43 as $8·6\times5$, and put $8·6=a+b$, and $5=a-b$, we obtain $43=6·8^2-1·8^2$; which gives a good result if the half-inch or centimetre is used as a unit of length.

EXERCISES LVI.†

RIDERS.

1. Show how to describe a square equal to the sum of two given squares.

2. The square described on the diagonal of a given square is double the given square.

3. The sum of the squares on the diagonals of a rectangle is equal to the sum of the squares on its four sides.

4. State how to make a square whose area is five times that of a given square.

5. $ABCDE$ is a polygon such that BC is perpendicular to AB, CD to AC, and DE to AD. Prove that the square on AE is equal to the sum of the squares on the remaining sides.

6. If two opposite sides of a quadrilateral meet at right angles when produced the sum of the squares on the other two sides is equal to the sum of the squares on the diagonals.

7. ABC is a triangle having the side AB greater than the side AC, and AD is drawn perpendicular to BC; prove that BD is greater than CD, and the angle BAD greater than the angle CAD.

8. ABC is any triangle, and AD is drawn perpendicular to BC. Show that the difference of the squares on AB, AC is equal to the difference of the squares on BD, CD.

9. If A and B are fixed points find the locus of a point P which moves so that the difference of the squares on AP and BP is equal to the square on AB.

10. Prove that if the difference of the squares on two sides of a triangle is equal to the square on the third side, the triangle is right-angled.

11. The square on any line is four times the square on half the line. Hence the sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

12. ABC is a triangle right angled at A, and E and F are the middle points of AC and AB. Prove that the sum of the squares on BE, CF is equal to five times the square on EF.

13. $ABCD$ is a parallelogram in which the diagonal BD is at right angles to the sides AB and CD. Prove that the difference of the squares on the diagonals is equal to four times the square on AB.

---

† In these Exercises assume that, if $AB > AC$, then sq. on $AB >$ sq. on AC, and conversely. This is proved by superposition.

14. C is a given point in a given line AB. Find the locus of a point P which moves in such a way that sq. on PA − sq. on PB = sq. on CA − sq. on CB.

15. Two sides of a triangle contain (i) an obtuse angle, (ii) an acute angle. Prove that the sum of the squares on these sides is (i) less than, (ii) greater than, the square on the third side.

[Let ABC be the triangle and A the acute or obtuse angle. Make BAD a right angle and $AD = AC$. Join BD. Use T.6 ; RT.2.]

16. The sum of the squares on two sides of a triangle is (i) less than, (ii) greater than, the square on the third side. Prove that these two sides contain (i) an obtuse angle, (ii) an acute angle.

[Make the same construction as in 15 ; use T.6c ; RT.2.]

17. In the figure of Theorem RT.2 :—

(*a*) Prove BG, CH are parallel.

(*b*) FA, AK are in the same straight line.

(*c*) If FC and AD intersect in N, prove that the angle ANF is equal to the right angle FBA.

(*d*) Join GH, FD, KE and prove that the triangles GAH, FBD, and KCE are each equal to ABC. [Use Ques. 7, Exercises *XLIX*.]

18. In an equilateral triangle the sum of the squares on the three sides is equal to four times the square on the perpendicular from the vertex to the base.

19. In the quadrilateral $ABCD$, given that $\angle B = \angle D = R$, and $AB > AD$, prove that $CB < CD$.

20.* ABC is a triangle and D is the middle point of BC, prove that the sum of the squares on AB and AC is greater than equal to or less than four times the square on AD according as the angle at A is obtuse, right, or acute.

21.* ABC is an acute-angled triangle, and D, E, F are the middle points of the sides. Prove that the sum of the squares on AB, BC, CA is less than twice the sum of the squares on AD, BE, CF. [Use Ques. 20.]

CALCULATIONS.

Calculate the length of the hypotenuse of a right-angled triangle if the other two sides measure :—

22. 7″, 4″. **23.** 2 cms., ·5 cm. **24.** 20 miles, 8 miles.

Calculate the remaining side of a right-angled triangle, if the hypotenuse and one side measure respectively :—

25. 3″, 2″. **26.** 10 cms., 4 cms. **27.** 20 miles, 14 miles.

28. There are three towns P, Q, R. P is 20 miles N. of Q and R is 12 miles E. of Q. Calculate the distance of P from R?

29. A door is 6 ft. high and 3 ft. 6 in. wide. Calculate to the nearest inch the distance between two opposite corners.

30. A ship started from a point 100 miles N. of a certain port. After sailing for 130 miles it is W. of that port. Calculate its distance from the port.

31. Calculate the altitude of an isosceles triangle whose base measures 4″ and whose equal sides measure 5″ each.

32. Calculate the base of an isosceles triangle whose equal sides measure 2″ each, and whose altitude measures 1·5″.

33. How many triangles can be formed by taking any three lines from lines 2, 3, 4, 5, 6, 7 inches long? Which of the triangles so formed will be right-angled?

34. ABC is a triangle right-angled at A; AD is drawn $\perp BC$, and $DF \perp AB$. If $AB = 4''$, $AC = 3''$, calculate the lengths BD, DC, AD, DF, BF, FA.

35. The sides of a triangle are each 10″, and its base is 7″. Calculate its altitude, and hence its area.

36. Calculate the altitude and area of an equilateral triangle whose side measures 12 yards.

37. Calculate the length of the side of an isosceles right-angled triangle whose base measures 2″.

38. The equal sides of an isosceles triangle measure 10″ each. Find the altitude if the base is 6″; and find the base if the altitude is 6″.

39. The hypotenuse of a right-angled triangle measures 8″ and one side measures 3″. Calculate the length of the other side and the area of the triangle.

40. In a trapezium the shorter of the two parallel sides measures 2″, and the two non-parallel sides each measure 1″. If the perpendicular distance between the parallel sides is ·7″, calculate the length of the remaining side and the area of the figure.

41. In the triangle ABC, the angle A is a right angle, and the sides AB, AC measure 2″ and 3·5″ respectively. Calculate the length of the hypotenuse and the area of the triangle. From these two results calculate the length of the line drawn from A perpendicular to BC.

42. In the quadrilateral $OBAC$, $\angle B = \angle C = \text{R}$, $OA = 3''$, $OB = OC = 1''$ Calculate the lengths AB, AC, the area $OBAC$, and the length BC.

43. Show algebraically that if the angles of a triangle are 90°, 45° 45°, the sides are in the proportion $\sqrt{2} : 1 : 1$.

44. Show algebraically that in a "half-equilateral" triangle the sides are in the proportion $2 : \sqrt{3} : 1$.

45. Calculate algebraically the perimeter of an equilateral triangle whose area is 2 square cms.

46. Calculate algebraically the perimeter of a triangle whose sides are in the ratio 3 : 4 : 5, and whose area is 1 sq. mm.

BY DRAWING AND CALCULATION.

47. Draw a square equal to the sum of two squares whose sides are respectively 1″, 2″. Measure its diagonal, and verify the result by calculation.

48. Draw a square equal to the difference of two squares whose sides are 22 and 33 mm. Measure its diagonal, and verify the result by calculation.

Evaluate by the method of § 157.

49 $\sqrt{13}$. **50** $\sqrt{5}$. **51.** $\sqrt{41}$. **52.** $\sqrt{29}$. **53.** $\sqrt{23}$.
54. $\sqrt{33}$. **55.** $\sqrt{43}$. **56.** $\sqrt{149}$. **57.** $\sqrt{1 \cdot 64}$. **58** $\sqrt{521}$.
59. $\sqrt{1 \cdot 25}$.

60 $19 = 4^2 + 1^2 + 1^2 + 1^2$. Hence and from Ques. 5 find $\sqrt{19}$.

CHAPTER X.

THE CIRCLE: SYMMETRY.

159. The Circle.

DEFINITIONS.—**A *circle*[†] is a plane figure bounded by one curved line called its circumference,[‡] which is such that all points on this circumference are at equal distances from a certain point which is called the centre[§] of the circle.**

Any line joining the centre to a point on the circumference is called a *radius*.[||]

Any line passing through the centre and terminated at both extremities by the circumference is called a *diameter*.[¶]

Any diameter divides a circle into two halves, each of which is called a *semi-circle*.[††]

Any portion of the circumference of a circle is called an *arc*.[‡‡]

Thus in Fig. 189 *ACB, ADB* are both arcs of the circle.

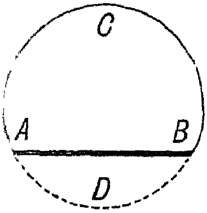

FIG. 239.

Any straight line joining two points on the circumference of a circle is called a *chord*.[§§]

Thus in Fig. 239 *AB* is a chord of the circle.

160. Symmetry.—Revise § 28 of the Introductory Course, pp. 50–52.

DEFINITION.—**If a line can be found on a figure such that when the figure is folded across that line the portion of the figure on one side of the line coincides with the portion on**

[†] Derived from a Latin word meaning a *ring;* compare *circus.*

[‡] From a Latin word meaning the *path round* (the circle).

[§] From a Greek word meaning originally a *prick.*

[||] From a Latin word meaning a *ray;* the radii pass outwards from the centre like rays of light from a flame.

[¶] From a Greek word meaning the *measure* (of the distance) *through* (the circle).

[††] *Semi* means *half* (Latin).

[‡‡] From a Latin word meaning *bow;* compare *archer, arch.*

[§§] The same word as *cord;* if the dotted arc in Fig. 189 is regarded as a bow strung ready for use, then the chord *AB* represents the string.

the other side, line for line and point for point; then the figure is said to be symmetrical about this line, and this line is called the axis of *symmetry.*†

DEFINITION.—**If a figure is symmetrical with regard to some line as axis then either half of the figure may be called the** *image* **of the other half in that line.**

In Fig. 244 if a plane metal‡ mirror is placed standing upright on the line *XY* and facing to the left, then the mirror would show an image (or "reflection") of the figure *APD* which would appear in the position *CPB.*

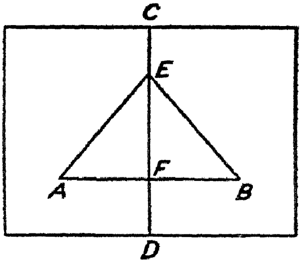

FIG. 240.

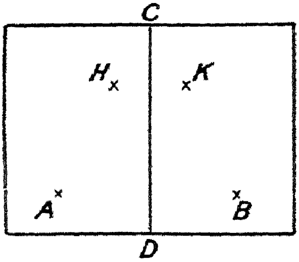

FIG. 241.

EXERCISES LVII (Practical).

1. Take a piece of paper and fold it into two; make one prick with a pin through both folds of the paper. Now unfold the paper; let *CD* be the crease, and *A, B* the two holes made by the pin (Fig. 240). Obviously *A, B* are symmetrical with regard to the crease *CD.*

Join *AB* cutting *CD* in *F.* Join *A* and *B* to any other point *E* on *CD.*

Measure *AF, BF, AE, BE,* ∠*AFC,* ∠*AEF,* ∠*BEF.* Mark each measurement on the figure. What do you infer?—(i) With regard to the lines joining two symmetrical points to any point on the axis of symmetry. (ii) With regard to the join of any two symmetrical points.

2. Take another piece of folded paper, and make *two* pricks with a pin through both folds of the paper, so that when the paper is unfolded you have a figure like that shown in Fig. 241; *A, B* and *H K* are symmetrical pairs of points with respect to the crease *CD.*

Join the lines *AH, BK, AK, BH.* What do you notice with regard to the following lines?— (i) *AK, BH, CD.* (ii) *AH, BK, CD.*

---

† The idea of folding a plane figure about a straight line is logically justified by dividing the figure into two parts by a cut along the line, and applying one part to the other. Use Axiom 10, p. 74, and the Definition of straight line, p. 67.

‡ An ordinary piece of "looking-glass" does not give a good result, as in this case the actual mirror lies behind a certain thickness of glass, and this causes complications.

3. Draw a circle on a piece of paper, and draw any diameter of the circle. Fold the paper about this diameter as crease, so that the circle is on the outer faces of the folded paper. Prick through both folds of the paper at several points on the circumference of the semi-circle on the upper face. Unfold the paper.

What do you notice, and what do you infer?

161. THEOREM [Sy.1].—**If two points are symmetrical about a given axis, then (i) their join is bisected at right angles by this axis, and (ii) their joins to any point on the axis are equal, and form an angle which is bisected by the axis.**

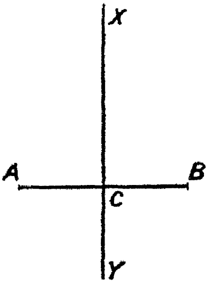

FIG. 242.

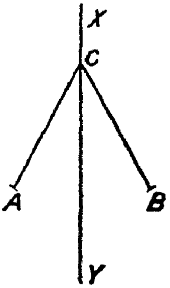

FIG. 243.

Given that the points A, B are symm. to the axis XY,
Required to prove that

(i) AB is bisected at right angles by XY (Fig. 242).
and (ii) $AC = BC$, and XY bisects $\angle ACB$ (Fig. 243).

Proof.—In each case suppose the figure folded about XY.

In each case, $\because$ A is symm. to B, $\therefore$ A falls on B; also in each case C coincides with itself.

Hence in each case the line AC coincides with the line BC, and $\angle ACY$ with $\angle BCY$.

Thus $\qquad AC = BC$, and $\angle ACY = \angle BCY$.

Also in Case (i), ACB is a straight line;
hence $\qquad XY \perp AB$.

COROLLARY.—**The re-entrant angle ACB is also bisected by the axis XY** (Fig. 243).

162. Theorem [Sy. 1c.].—**Two points are symmetrical to a given axis if (i) their join is bisected at right angles by this axis, or (ii) their joins to some point on the axis are equal, and form an angle which is bisected by that axis.**

Given that
(i) AB is bisected at right angles by XY, (Fig. 242), or
(ii) $AC = BC$, and XY bisects $\angle ACB$ (Fig. 243).
Required to prove that
the points A, B are symmetrical to the axis XY.

Proof.—In each case suppose the figure folded about the line XY.

In each case $\angle ACY = \angle BCY$; $\therefore$ CA will lie along CB.
In each case $CA = CB$; $\therefore$ A will fall on B.

Hence A and B are symmetrical about the axis XY.

163. Theorem [Sy. 2].—**A circle is symmetrical about any diameter.**

Required to prove that $\odot$ PXQ (Fig. 244) is symmetrical to the diameter XY; *i. e.* that XY divides the circumference into two symmetrical portions.

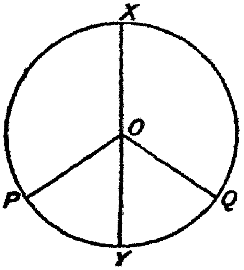

Fig. 224.

Cons.—Take any point P on the $\bigcirc^{ce}$, and join it to the centre O.

On the opposite side of XY draw a radius OQ, making $\angle QOY = \angle POY$.

Proof.— $\because$ radius OP = radius OQ, and $\angle POY = \angle QOY$;
$\therefore$ P is symmetrical to Q with respect to XY. Sy. 1c.

Similarly any other point on either† portion of the $\bigcirc^{ce}$ is symmetrical to a point on the other portion.
Hence the two portions of the $\bigcirc^{ce}$ are symmetrical.

---

† Note the importance of the word "either." It is necessary to show that any point on the *left* side is symmetrical to some point on the right side, and *also* that any point on the right side is symmetrical to some point on the left side.

164. THEOREM [Sy.3].—**If a pair of points is symmetrical to a second pair, then the joins of these pairs either are parallel to, or meet on, the axis of symmetry.**

Given that A, B are respectively symmetrical to C, D with respect to the axis XY (Figs. 245, 246).

Required to prove that AB and CD are either parallel to XY or else intersect on XY.

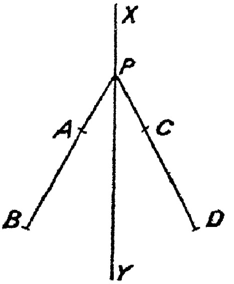

FIG. 245.

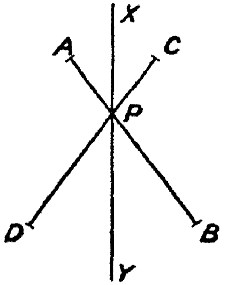

FIG. 246.

CASE I.—A and B on the same side of XY (Fig. 245).

Proof.—Suppose the figure folded over the line XY.

$\because A, B$ are symm. to C, D, $\therefore A$ falls on C, and B on D.

Hence the straight lines AB and CD coincide.

Also the line in which AB, CD coincide is either parallel to the crease or intersects it.

Hence AB and CD either are both parallel to XY, or else meet XY at the same point.

CASE II.—A and B on opposite sides of XY (Fig. 246).

Cons.—Let the st. line AB meet XY at P. Join CP, DP.

[Required to prove that CPD is a st. line.]

Proof.—$\because A$ is symm. to C, $\therefore \angle APX = \angle CPX$. Sy.1.

$\because B$ is symm. to D, $\therefore \angle BPX = \angle DPX$. Sy.1.

Hence $\angle APX + \angle BPX = \angle CPX + \angle DPX$.

$\because APB$ is a straight line, $\therefore \angle APX + \angle BPX = 2\text{R}$; A.1.

Hence $\angle CPX + \angle DPX = 2\text{R}$.

$\therefore CPD$ is a straight line; A.1c.

Hence the straight lines AB and CD meet on XY.

COROLLARY.—**Symmetrical lines either are parallel to, or meet on, the axis of symmetry.**

For symmetrical lines will always join two pairs of symmetrical points.

165. **Argument by Symmetry.**—Theorems Sy.1*c* and Sy.2 are *tests* of symmetry. Theorems Sy.1 and Sy.3 are simple and useful results which follow from symmetry. All other arguments by symmetry depend on the fact that when a symmetrical figure is folded about the axis of symmetry *the two parts become identical.*

The following examples will illustrate the use of symmetry in logical argument, and will also serve as models of the form in which such arguments should be stated.

Example 1.—*Prove by symmetry that a diameter of a circle bisects any chord to which it is perpendicular.*

·In ⊙ ABC (Fig. 247) ;—
Given that diam. $CD \perp$ chord AB,
Required to prove that $EA = EB$.

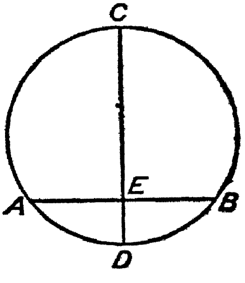

FIG. 247.

Proof.—Suppose the figure folded across the line CD.
$\because CD$ is a diam., $\therefore$ arc CAD coincides with arc CBD. Sy.2.
$\because \angle CEA = \angle CEB$, $\therefore$ the line EA lies along the line EB.

Thus the point A, where the line EA meets the arc CAD, coincides with the point B, where the line EB meets the arc CBD.

Hence $EA = EB$.

NOTE.—In this proof we assume that the line AB does not meet the circle in more than two points. If it did, A might not coincide with B but with one of the other points of intersection.

Example 2. —*The hexagon ABCDEF is symmetrical about the diagonal AD* (Fig. 248). *Prove that* (i) *the two lines BP, FQ, which bisect the angles B, F, either are parallel or meet on AD, and* (ii) *the two lines HK, LM, which bisect the sides BC, EF at right angles, either are parallel or meet on AD.*

Proof.—(i) ∵ *BP, FQ* bisect the symm. angles *B, F,*
∴ *BP, FQ* are symm. about *AD;*

for otherwise, when the figure is folded about *AD,* we should have two different lines bisecting the same angle *F*—which is not possible.

Hence *BP, FQ* either are parallel or meet on *AD.* Sy.3.

ii) ∵ the points *H, L* bisect the symm. lines *BC, FE,*
∴ *H, L* are symm. about *AD;*

for otherwise, when the figure is folded about *AD,* we should have two different points bisecting the same line *FE*—which is not possible.

∵ *HK, LM* are drawn from the symm. points *H, L,* and are perpendicular to the symm. lines *BC, FE,*

∴ *HK, LM* are symm. about *AD;*

for otherwise, when the figure is folded about *AD,* we should have two different lines perpendicular to the same line *FE* at the same point *H*—which is not possible.

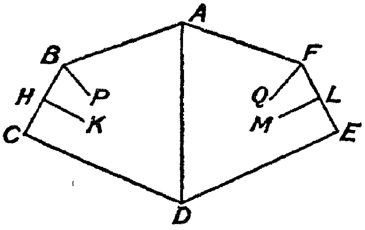

FIG. 248.

Hence *HK, LM* either are parallel or meet on *AD.*

166.* **Point-Symmetry.**—§§ 160–165 deal with symmetry with respect to a line or "axis." There is also another kind of symmetry, viz. symmetry with respect to a point.

DEFINITION.—**Two points are said to be *symmetrical* with respect to a third point if the join of the first two points is bisected by the third point.**

Thus in Fig. 249 the points *P* and *p* are symmetrical with regard to the point *O*; for *O* bisects the line *Pp*.

DEFINITION.—**A figure is said to be** *symmetrical with respect to a given point*, **if each point on the figure is symmetrical to some other point on the figure with respect to the given point.**

Thus in Fig. 249 the curve *PQpq* is symmetrical with respect to the point *O*; for *P* is symmetrical to *p*, *q* to *Q*, *S* to *s*, *a* to *A*, and so on.

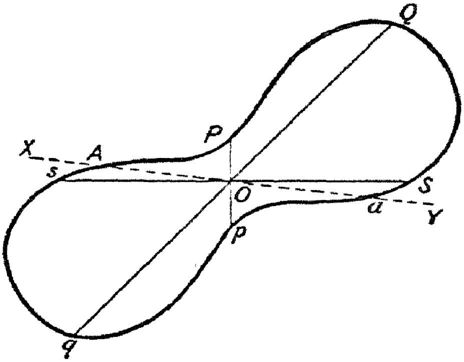

FIG. 249.

Suppose Fig. 249 to be divided into two parts by any straight line *XY* passing through *O*. Then the part *APQSa* will be congruent to the part *apqsA*.

For if the part *APQSa* is swung round *O* till *OA* falls on *Oa*, then *A* falls on *a*; for $OA = Oa$.

Also, since $\angle AOQ = \angle aoq$ (A.2), ∴ *OQ* falls along *Oq*; thus *Q* falls on *q*, for $OQ = Oq$.

Similarly *P* falls on *p*, *S* on *s*, etc., and thus the two portions of the figure are congruent.

EXERCISES LVIII (Riders).

1.† Prove that a straight line cannot meet a circle in more than two points. T.1, T.2.

2. *AB, CD* are two chords of a circle perpendicular to a given diameter, *A* and *C* being on the same side of this diameter. Prove that the chords *AC* and *BD* are equal, and that they intersect on the given diameter, unless they are parallel. Sy.2; and folding.

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† Assume this result in proving Questions 3, 5, 11, 12.

3. From one extremity of a diameter two chords are drawn equally inclined to that diameter. Prove that the chords are equal in length. Sy. 2; and folding.

4. *C* is the centre of a circle, *O* is any given point, and *A* any point on the circumference. Show that it is always possible to find a second point *B* on the circumference, such that $OB = OA$, unless *A*, *C* and *O* lie in a straight line.

5. Any chord divides the circumference of a circle into two arcs. Show that the diameter perpendicular to this chord bisects each of these arcs.

6. Given two circles show that there is always one line about which both circles are symmetrical. When is there more than one such line?

7. Two circles whose centres are *A*, *B* meet at two, and only two, points *C*, *D*. Show that *AB* bisects *CD* at right angles.

8.* Show that the three lines which bisect the angles of an isosceles triangle are concurrent. Folding, Sy. 3.

9.* Prove that an isosceles triangle is symmetrical about the line which joins the vertex to the mid-point of the base. Hence show that the three lines which join the angular points of an isosceles triangle to the mid-points of the opposite sides are concurrent. Sy. 1*c*, Sy. 3.

10.* Show that a rectangle is symmetrical about the line which joins the mid-points of a pair of opposite sides. Hence show that this line passes through the intersection of the diagonals of the rectangle Pm. 2, Sy. 3.

11.* *O* is a point outside a circle whose centre is *C*. Two lines *OPQ*, *ORS* make equal angles with *OC*, and meet the circumference at the points *P*, *Q*, *R*, *S*. Prove that *PR*, *QS* are parallel, and that *OC*, *PS*, *QR* are concurrent.

12.* *AB*, *CD* are two chords of a circle; *OE*, *OF* are the perpendiculars drawn to these chords from the centre *O*. If $OE = OF$, prove that $AB = CD$. [Use an axis of symmetry bisecting $\angle EOF$].

13.* Prove that a circle is symmetrical with respect to its centre.

14.* What symmetry is possessed by the following capital letters

A, D, M, O, X, Z ?

15.* Prove that a parallelogram is symmetrical with regard to the middle point of either diagonal. Deduce that the two diagonals bisect each other. P. 1*c*, C. 2.

CHAPTER XI.

THE CIRCLE: CHORDS AND TANGENTS.

167. DEFINITIONS.—**A *secant*† of a circle is a straight line which meets the circumference in two points.**

Thus in Fig. 250 the secant *EF* meets $\bigcirc^{ce}$ at the points *G* and *H*.

A *tangent*‡ to a circle is a straight line which meets the circle at one and only one point however far the line is produced.

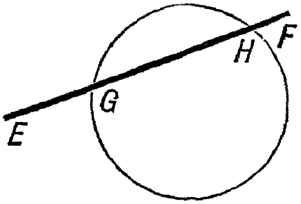

FIG. 250.

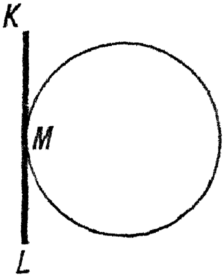

FIG. 251.

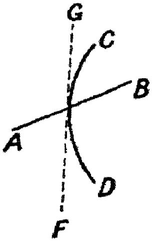

FIG. 252.

Any tangent is said to **touch** the circle; and the point where it meets the circle is called the **point of contact.**

Thus in Fig. 251 *KL* is a tangent which touches the circle at the point *M*; and *M* is called the point of contact.

A line is said to **cut** a second line if it passes through the second from one side to the other.

Thus in Fig. 252 the line *AB* cuts the arc *CD*.

Contrast this with the tangent *FG* which "grazes" the curve *without passing from one side of the curve to the other.*

168. **Alternative definition of tangent.**—The definition of a tangent given in § 167 has the disadvantage that it does not apply to all curves. For example in Fig. 253 the line *t* is the tangent at the point *P* to the curve *CD*; for at *P* the line "grazes" the curve without passing through it from one side to the other; but *t* also meets the curve at *K*, and hence *cannot* be described as

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† From a Latin word meaning *cutting*; compare the word *section*.

‡ From a Latin word meaning *touching*.

a line which meets the curve in "one and only one point." There is, however, another method of approaching the subject.

In Fig. 253 the line s meets the curve at the points P and Q (and another point), and is called a **secant**.

DEFINITION.—**Any straight line which meets a curve at two or more points is called a** *secant.*

Now suppose the point Q to move along the curve in the direction of P, passing through the positions Q_1, Q_2, . . .; then the secant passes through the positions s_1, s_2, . . ., and as Q approaches nearer to P, the line s approaches nearer to the position of the line t.

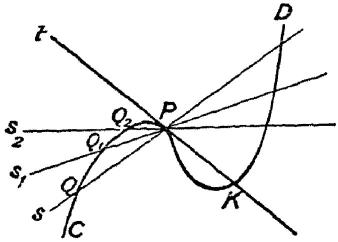

FIG. 253.

If the point Q moves on till P and Q become **consecutive points**, *i.e.* points with absolutely no distance between them, the secant then takes the position t and becomes the tangent.† Thus the tangent at P may be regarded as the "limiting" position of the secant through P, when a second point of intersection of the secant with the curve moves up to P.

DEFINITION.—*A tangent* **to a curve is a line which meets the curve in two consecutive points.**

This definition leads to valuable illustrations of the properties treated in this chapter.

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† To illustrate this drive a pin through the paper at P, and place the straight edge against the pin so as to show the line of the secant s. Then turn the straight edge gradually round the pin through the positions s_1, s_2, to the position t.

A more exact study of the many difficulties involved in this definition is needed before these illustrations can be regarded as *proofs*, and this must be deferred to a later stage.

EXERCISES LIX.

1. Draw a circle of radius about 2″. Draw any chord AB, not passing through the centre; bisect it at E, and through E draw a diameter CD. Measure $\angle CEA$.

Repeat the whole construction three times, varying the position of AB. What do you infer?

2.* Draw a circle of radius about 2″. Take two points A, B on the circumference about 0·3″ apart. Draw the secant through these two points, and bisect the line AB at E; through E draw a diameter CD. Measure $\angle CEA$. You should obtain the same result as in Question 1.

Imagine the two points where the secant meets the circumference to coalesce, and the result of Question 1 to be still correct.

What do you infer?

3. Draw a circle of radius about 1″. Draw a tangent to the circle. Note that it is easy to draw a fairly good tangent to the circle, but not easy to tell the exact point of contact.

Now draw, *as accurately as you can*, a line from the centre perpendicular to the tangent.

What do you notice? How would you use this result to draw the tangent at some *given* point on the circumference?

4. Draw a circle. Draw five or six chords in the circle all parallel to each other. Bisect each chord.

What do you notice about the mid-points of the chords? What locus-theorem would you infer from this result?

5*. In the figure of Question 4, draw accurately a secant which is parallel to any of these chords, and whose two points of intersection are close together.

Does the locus-theorem still hold?

Suppose the secant to become a tangent parallel to any of these chords; what do you now infer?

6*. O is the centre of a circle, and $ABCD$ a secant which meets the circle at the points B, C.

Calculate the magnitude of the angle OCD, if the magnitude of the angle BOC is (i) 20°, (ii) 8°, (iii) 2°, (iv) 0·2°, (v) 0·002°. What do you infer?

7. Draw a circle of radius about 1·2″. Draw six or eight chords each of length 1″, and mark (roughly) their middle points.

What locus-theorem does the figure suggest?

8. If the radius of a circle is 1″, what is the length of the greatest chord which can be drawn in the circle?

9. If two chords of a circle are of unequal length, which of the two would you expect to be nearer to the centre?

169. Theorem [Ch.1].—**If a diameter of a circle bisects a chord which is not a diameter, it is perpendicular to that chord; and conversely if the diameter is perpendicular to the chord it bisects it.**

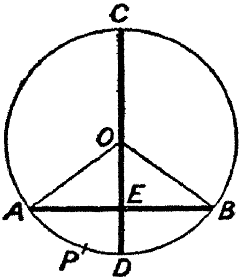

Fig. 254.

(i) In the circle ABC (Fig. 254):—

Given that the diameter CD bisects the chord AB,

Required to prove that $CD \perp AB$.

Cons.—Join the centre O to the points A and B.

Proof.—In $\triangle$s $OEA, OEB,$

$$\because \left\{\begin{matrix} OA = OB, \\ OE = OE, \\ AE = BE, \end{matrix}\right\} \therefore \angle OEA = \angle OEB. \qquad \text{C.3.}$$

But these are adj. angles; **hence** $CD \perp AB$.

(ii) In the circle ABC (Fig. 254):—

Given that the diameter $CD \perp$ the chord AB.

Required to prove that CD bisects AB.

Cons.—Join the centre O to the points A and B.

Proof.—In $\triangle$s $OEA, OEB,$

$$\because \left\{\begin{matrix} OA = OB, \\ OE = OE, \\ \angle OEA = \angle OEB = \mathrm{R}, \end{matrix}\right\} \therefore AE = BE; \qquad \text{C.4.}$$

hence CD bisects AB.

Corollary 1.—**A line cannot meet a circle in more than two points.**

For if the line AB (Fig. 254) meets the circle at some third point K, and if OE is drawn $\perp$ this line from the centre O, then E would bisect AB and (by a similar argument) would also bisect AK; which is impossible.

Corollary 2*.—[Tn.1] **The radius through any point is perpendicular to the tangent at that point.**

Let P be the given point (Fig. 254).

We know that so long as A and B are any two points on ○ce and E bisects AB, then $OE \perp AB$. *Proved.*

Let A and B both move up to and coincide with P.

Then the line AB becomes the tangent at P.

But in this case E will also coincide with P; thus OE becomes the radius OP.

Hence OP is perpendicular to the tangent at P.

NOTE.—The second part of the preceding theorem gives an alternative proof that the circle is symmetrical to any diameter.

170. THEOREM [Ch.2].—**The line which bisects any chord of a circle at right angles passes through the centre.**

In the circle ABC (Fig. 255):—

Given that CD bisects the chord AB at right angles,

Required to prove that CD passes through the centre.

Proof.— $\because CD$ bisects AB at right angles,

$\therefore CD$ is the locus of *all* points equidistant from A and B. L.3.

But the centre is equidistant from A and B;

hence the line CD passes through the centre.

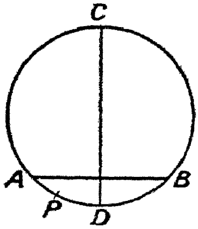

FIG. 255.

COROLLARY*.—[Tn.1] **The line drawn perpendicular to tangent at its point of contact will pass through the centr**

Let P be the point of contact of the tangent (Fig. 255).

We know that so long as A and B are any two points on the circumference, the line which bisects AB at right angles passes through the centre. *Proved.*

Let A and B move up to and coincide with P.

Then the line AB becomes the tangent at P.

Also the middle point of AB coincides with P.

Hence the line which bisects AB at right angles becomes the line perpendicular to the tangent at P.

Hence this perpendicular passes through the centre.

NOTE.—Theorems Ch.1 and Ch.2 are obviously related; but one is not the converse of the other (see § 59). There is a group of three facts connected with the line CD, viz.—

(*a*) It passes through the centre;
(*b*) It bisects the chord AB;
(*c*) It is perpendicular to the chord AB.

It follows from Ch.1 and Ch.2 that *given any two* of these facts we can *prove* the third.

171. THEOREM [Ch.3].—**It is possible to draw one and only one circle through three points which do not lie in a straight line.**

Given that the three points A, B, C do not lie in a straight line (Figs. 256 and 257),

Required to prove that it is possible to draw one and only one circle to pass through A, B and C.

Cons.—Join AB, BC.

Bisect AB and BC at right angles by the lines DF, EG.

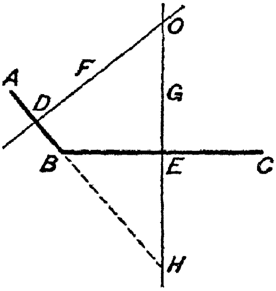

FIG. 256.

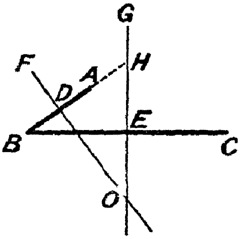

FIG. 257.

Proof.—(i) To prove that DF and EG are not paral

If possible let DF be $\parallel$ EG.

Then no other line through D can be $\parallel$ EG; *Axiom.*
hence AB meets EG at some point H.

$\because DF \perp BH$, and $EG \parallel DF$, $\therefore EG \perp BH$. P.2*c*.

Thus, in $\triangle BEH$, $\angle E$ and $\angle H$ are both right angles.
But this is impossible. T.1.

Hence DF and EG are not $\parallel$ but meet at some point O.

(ii) To prove that one circle can be drawn through A, B, C:—

$\because$ DF bisects AB at right angles, $\therefore$ $OA = OB$. L.3.
$\because$ EG bisects BC at right angles, $\therefore$ $OB = OC$. L.3.

$\because$ $OA = OB = OC$, a circle drawn with centre O and radius OA will pass through the three points A, B, and C.

Hence one circle can be drawn through A, B, C.

(iii) To prove that no other circle can be drawn through A, B, and C:—

Let P be the centre of *any* circle through A, B, C.

$\because$ $PA = PB$, $\therefore$ P lies on DF; L.3.
$\because$ $PC = PB$, $\therefore$ P lies on EG. L.3.

Hence P can only lie at the one point O where DF and EG intersect.

Thus the centre of the circle must lie at O, and the radius must be OA.

Hence only one circle can be drawn through A, B, C.

COROLLARY.—**Two circles cannot meet in more than two points.** •

For two different circles cannot pass through the same three points.

EXERCISES LX (Riders).

1. Through a given point on the circumference of a circle, two equal chords are drawn; show that they make equal angles with the line drawn from the given point to the centre.

2.•AB, AC, are two equal chords of the circle ABC; show that the straight line which bisects the angle BAC also passes through the centre of the circle ABC. Ch.2.

3. The angular points of a rectilineal figure all lie on a circle. Prove that the straight lines which bisect the sides at right angles all pass through a fixed point. Ch.2.

4. AB is a chord of a circle, and C, D are points in which the chord is cut by a circle with the same centre. Prove that $AC = DB$.

5. Show that the locus of the middle points parallel chords in a circle is a straight line. Ch.1.

6. Find the locus of the centre of a circle which passes through two given points.

172. **Congruent Circles.**—It is scarcely necessary to prove that any two circles of equal radii (x and y, in Fig. 258) are congruent. Place the one circle x on the other y so that A, the centre of x, falls on B, the centre of y. Then if P is any point on $\bigcirc^{ce}$ of x,

distance of P from B

$$\begin{aligned} &= \text{distance of } P \text{ from } A \\ &= \text{radius of } x \\ &= \text{radius of } y. \end{aligned}$$

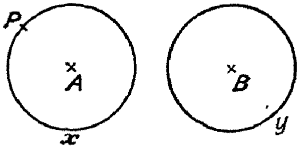

FIG. 258.

Hence P lies on $\bigcirc^{ce}$ of y.

Similarly any point on $\bigcirc^{ce}$ of y falls on $\bigcirc^{ce}$ of x. Hence the two circles are congruent.

The converse proposition is equally obvious, viz. that if two circles are congruent their radii are equal. For when the two circles coincide the centre of one circle is obviously *a* centre of the other; and it is easy to show that the second circle cannot have two different centres.

173. THEOREM [Ch.4].—**In any circle equal chords are equidistant from the centre; and conversely, chords which are equidistant from the centre are equal.**

AB, CD are chords in a $\odot$, whose centre is O (Fig. 259).

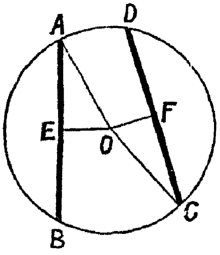

FIG. 259.

(i) *Given* that OE, OF are drawn $\perp$ the chords, and that $AB = CD$; *Required to prove* that $OE = OF$.

Cons.—Join OA, OC.

Proof.— $\because$ O is the centre,
and $OE \perp$ chord AB,
$\therefore$ E bisects AB. Ch.1.
Similarly F bisects CD.

But $AB = CD$; $\therefore$ their halves are equal, *i.e.* $AE = CF$.

$$\text{In } \triangle\text{s } AOE, COF, \because \left\{ \begin{array}{l} OA = OC, \\ AE = CF, \\ \angle OEA = \angle OFC = \text{R}, \end{array} \right.$$

therefore $OE = OF$ C.4.

(ii) *Given* that OE, OF are drawn perpendicular to the chords, and that $OE = OF$,

Required to prove that $AB = CD$.

Cons.—Join OA, OC.

Proof.—As before it can be proved that

E bisects AB, and F bisects CD. Ch. I.

In $\triangle$s AOE, COF, $\because \left\{\begin{array}{c} OA = OC, \\ OE = OF, \\ \angle OEA = \angle OFC = \text{R}, \end{array}\right\} \therefore EA = FC$. C. 4.

But AB, CD are double of EA, FC respectively.

Hence $AB = CD$.

COROLLARY.—**In equal circles equal chords are equidistant from the centres; and conversely, chords which are equidistant from the centres are equal.**

174. THEOREM [Ch. 5].—**In any circle, if two chords are equal they subtend equal angles at the centre; and conversely, chords which subtend equal angles at the centre are equal.**

In $\odot\ ABC$ (Fig. 260):—

(i) *Given* that chord AB = chord CD,

Required to prove that $\angle\alpha = \angle\delta$.

Proof.—In $\triangle$s OAB, OCD, $\because \left\{\begin{array}{l} OA = OC, \\ OB = OD, \\ AB = CD, \end{array}\right.$

therefore $\angle\alpha = \angle\delta$. C. 3.

(ii) *Given* that $\angle\alpha = \angle\delta$,

Required to prove that chord AB = chord CD.

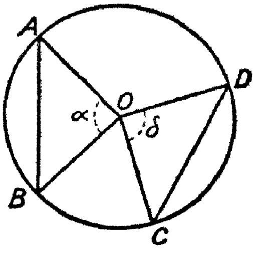

FIG. 260.

Proof.—In $\triangle$s OAB, OCD, $\because \left\{\begin{array}{l} OA = OC, \\ OB = OD, \\ \angle\alpha = \angle\delta, \end{array}\right.$

therefore $AB = CD$. C. 1.

COROLLARY.—**In equal circles, if two chords are equal they subtend equal angles at the centres; and conversely, chords which subtend equal angles at the centres are equal.**

175. THEOREM [Ch. 6].—**In any circle the diameter is the greatest chord. And of any other chords a chord which is nearer to the centre is greater than one which is more remote; and conversely a chord which is greater than another is nearer to the centre.**

(i) *Given* that AB is a diam. of the circle ABC (Fig. 261), *Required to prove* that $AB >$ any other chord CD.

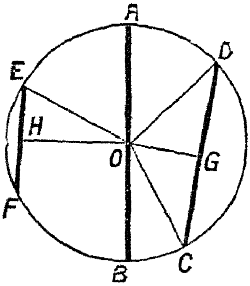

FIG. 261.

Cons.—Join the centre O to C, D.

Proof.—$OA = OD$, and $OB = OC$; add the equals, thus $AB = OC + OD$.

In $\triangle\ COD$, $OC + OD > CD$. T. 4.

Hence $AB > CD$.

(ii) CD, EF are chords of a circle whose centre is O, (Fig. 261) and OG, $OH \perp$ these chords.

Given that $OG < OH$,

Required to prove that $CD > EF$.

Proof.—In $\triangle$s OGD, OHE, $\because \angle G = \angle H = R$,

$\therefore$ sq. on OD = sq. on OG + sq. on GD, RT. 2.

and sq. on OE = sq. on OH + sq. on HE. RT. 2.

But $OD = OE$, hence sq. on OD = sq. on OE, Pm. 7.

$\therefore$ sq. on OG + sq. on GD = sq. on OH + sq. on HE.

Again $OG < OH$ (*Hyp.*), hence sq. on $OG <$ sq. on OH;†

$\therefore$ sq. on $GD >$ sq. on HE, whence $GD > HE$.‡

$\because OG$, $OH \perp$ chords CD, EF respectively,

$\therefore CD = 2GD$, $EF = 2HE$. Ch. 1.

But $GD > HE$ (*Proved*); **hence** $CD > EF$.

(iii) *Conversely.*—*Given* that $CD > EF$ (Fig. 261), *Required to prove* that $OG < OH$.

Proof.—As in (ii), $CD = 2GD$, $EF = 2HE$.

But $CD > EF$; $\therefore GD > HE$; $\therefore$ sq. on $GD >$ sq. on HE.†

As in (ii), sqs. on OG, GD = sqs. on OH, HE;

$\therefore$ sq. on $OG <$ sq. on OH, **whence** $OG < OH$.‡

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† Usually assumed. Easily proved by superposition.

‡ Usually assumed. Easily proved by "method of exhaustio (see note on p. 130).

176. **Alternative Proof.**—The following proof of Parts (ii) and (iii) of Theorem Ch. 6 can be used by students who have not yet read Chapters VII, VIII, IX.

(ii) In Fig. 262 $OG < OH$. *Required to prove* $CD > EF$.
Cons.—Join OD, OE, OF. On opposite side of OD to G, draw radius OK, making $\angle DOK = \angle EOF$. Join DK. Draw $OL \perp DK$. Join GL.

Proof.—$\because \angle DOK = \angle EOF$, $\therefore$ chord $DK =$ chord EF (Ch. 5); $\therefore OL = OH$ (Ch. 4).
But $OG < OH$; $\therefore OG < OL$.
Hence in $\triangle OGL$, $\angle\beta < \angle\alpha$. T. 3.
But $\angle\beta + \angle\delta = R = \angle\alpha + \angle\gamma$.
Thus $\angle\delta > \angle\gamma$; $\therefore$ in $\triangle DLG$, $DG > DL$. T. 3c.
$\because OG, OL \perp$ chords CD, DK,
$\therefore CD = 2DG$, $DK = 2DL$. Ch. 1.
Thus $CD > DK$, **that is** $CD > EF$.

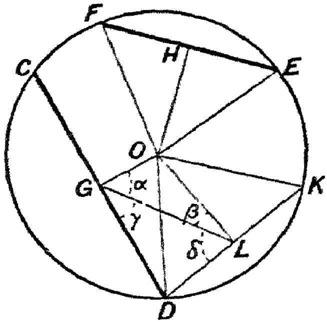

Fig. 262.

(iii) In Fig. 262, $CD > EF$.
Required to prove that $OG < OH$.
Cons.—The same as in (ii).

Proof.—As in (ii) $DK = EF$, $OL = OH$; $CD = 2DG$, $DK = 2DL$.
$\because CD > EF$, $\therefore CD > DK$; thus $DG > DL$.
Hence, in $\triangle DLG$, $\angle\delta > \angle\gamma$ T. 3.
But $\angle\beta + \angle\delta = \angle\alpha + \angle\gamma$; $\therefore \angle\beta < \angle\alpha$.
Thus, in $\triangle OGL$, $OG < OL$. T. 3c.
But $OL = OH$, **hence** $OG < OH$.

DEFINITION.—**The angle subtended by a given line at a given point is the angle traced out by a line which joins the given point to a moving point, when the latter travels along the given line from one extremity to the other.**

Thus in Fig. 254 the angle subtended at O by the straight line AB is AOB; and the angle subtended at O by the arc ADB is also AOB; and the angle subtended at O by the arc ACB is the re-entrant angle AOB.

EXERCISES LXI (Riders).

1. Given two circles and a chord in each; if the circles are equal and the chords are equal prove that the two figures are congruent.

2. In any circle the greater chord subtends a greater angle at the centre than the less.

3. Of any two chords in a circle that which is nearer to the centre subtends the greater angle at the centre.

4. Find the locus of the middle points of equal chords of a circle.

177. THEOREM [Tn.1].—**The line drawn through any point on the circumference of a circle at right angles to the radius through that point is a tangent to the circle; and any other line drawn through that point is a secant.**

In Fig. 263 O is the centre of a circle and P is a point on the circumference;

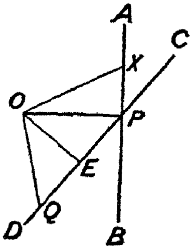

FIG. 263.

Given that $APB \perp OP$, and that CPD is not $\perp OP$,

Required to prove that (i) APB is a tangent to the circle, and (ii) CPD is a secant.

(i) *Cons.*—Join O to any point X in AB.

Proof.—Since $OP \perp AB$, $\therefore OX > OP$; T.5.

i.e. OX is greater than the radius of the circle.

$\therefore X$ does not lie on the $\bigcirc^{ce}$ but lies outside it.

Similarly it can be shown that every point on the line AB, excepting P, lies outside the $\bigcirc^{ce}$; *i.e.* AB meets the circumference only at the point P.

Hence AB is a tangent to the circle.

(ii) *Cons.*—Draw $OE \perp CD$ and mark off from PE produced a part EQ equal to EP. Join OQ.

Proof.—$\because EO$ bisects PQ at right angles, $\therefore OQ = OP$. L.3.
Because OQ is equal to the radius of the circle, therefore Q lies on the $\bigcirc^{ce}$.

Hence CD meets the $\bigcirc^{ce}$ at the two points P, Q;
that is CD is a secant to the circle.

COROLLARY 1.—**Any tangent and the radius through its point of contact are perpendicular to each other.**

For any line which is not perpendicular to this radius is not a tangent, but a secant.

COROLLARY 2.—**The line drawn at right angles to a tangent at the point of contact passes through the centre of the circle.**

For the radius through this point is perpendicular to the tangent, and there cannot be two different lines perpendicular to the tangent at the point of contact.

NOTE.—This proposition is only a recapitulation of the second Corollary to Theorem Ch.1, and the Corollary to Theorem Ch.2. The proposition is however too important to be treated merely in corollaries, and moreover the proof here given is valuable as it dispenses with the difficult idea of consecutive points on a curve.

178. THEOREM [Tn.2].—**If two tangents are drawn to a circle from the same point outside the circle, they are equal in length; also the line joining this point to the centre bisects the angle between the tangents, and the angle between the radii to the points of contact of the tangents, and bisects at right angles the line joining the two points of contact.**

Given that PL, PM are tangents to the circle whose centre is O, and that L, M are the points of contact (Fig. 264),

Required to prove that $PL = PM$, and that PO bisects the angles LPM and LOM, and bisects at right angles the line LM.

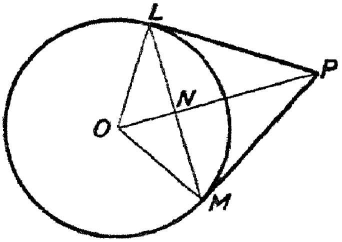

FIG. 264.

Proof.—$\because$ OL, OM are radii and PL, PM tangents;

$\therefore \angle OLP = \angle OMP = \text{R}.$ Tn.1.

$$\text{In } \triangle\text{s } OLP, OMP, \because \begin{cases} OP = OP, \\ OL = OM, \\ \angle OLP = \angle OMP = \text{R}, \end{cases}$$

therefore $PL = PM$, $\angle OPL = \angle OPM$, $\angle LOP = \angle MOP$. C.4.

$$\text{In } \triangle\text{s } LON, MON, \because \begin{cases} ON = ON, \\ OL = OM, \\ \angle NOL = \angle NOM, \end{cases}$$

therefore $\left.\begin{array}{l} LN = MN. \\ \angle LNO = \angle MNO \end{array}\right\}$ C.1.

Also

But these are adj. angles; hence $OP \perp LM$.

COROLLARY 1.—**The whole figure is symmetrical about the line** OP.

COROLLARY 2.—**It is not possible to draw more than two tangents from any point to the circle.**

For if PK were another tangent to the circle drawn from P, and K the point of contact, then by the above argument OP would bisect each of the angles LOM, LOK, MOK; which is obviously impossible.

EXERCISES LXII (*Riders*).

1. If two tangents to a circle are parallel the line joining their points of contact passes through the centre.

2. Find the locus of the centre of a circle which touches a given line at a given point.

3. A circle of given radius rolls along a given straight line. Find the locus of the centre of the circle.

4. O is the centre of a circle and PQ is any chord. Prove that the angle between PQ and the tangent at P is half of the angle POQ.

5. Find the locus of the centre of a circle which touches the two arms of a given angle.

6. All chords of a circle which touch a concentric circle are bisected at the point of contact. Ch. 1.

7. All chords of a circle which touch a concentric circle are equal. Ch. 4.

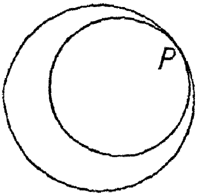

FIG. 265.

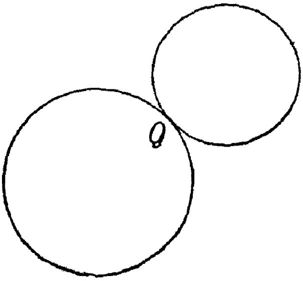

FIG. 266.

179. Circles touching circles.—

DEFINITION.—**Two circles are said to touch one another if they meet at one and only one point.**

The point at which the circles meet is called the **point of contact.** Thus in Figs. 265, 266, P and Q are the points of contact.

In Fig. 265 the two circles are said to touch *internally* at the point *P*; and in Fig. 266 the two circles are said to touch *externally* at the point *Q*.

We have seen in the Corollary to Theorem Ch.3 that two circles cannot meet at more than two points. If they meet at two points they are said to **cut** one another. The line joining these points is a chord of both circles and is therefore called their **common chord.**†

180. THEOREM [Ch.7].—**If two circles cut one another the line joining their centres bisects the common chord at right angles.**

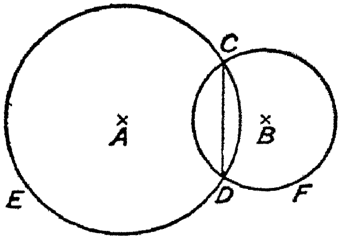

FIG. 267.

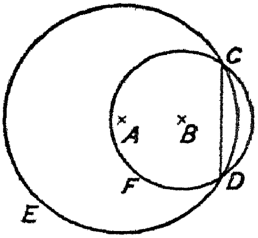

FIG. 268.

The two circles *CDE, CDF* (whose centres are *A, B*) intersect at the points *C, D* (Figs. 267, 268):—

Required to prove that *AB* bisects *CD* at right angles.

Cons.—Let *HK* be the line which bisects *CD* at right angles.‡

Proof.—

In ⊙ *CDE*, ∵ *HK* bisects the chord *CD* at right angles,
∴ *HK* passes through the centre *A*. Ch.2.
In ⊙ *CDF*, ∵ *HK* bisects the chord *CD* at right angles,
∴ *HK* passes through the centre *B*. Ch.2.
Thus *HK* coincides with the line *AB*.
Hence *AB* bisects *CD* at right angles.

---

† The definition given in § 179 may be replaced by the following:—

DEFINITION.—**Two circles which meet at two consecutive points are said to *touch* one another.**

Such circles cannot meet in any other point, for two circles cannot meet in three points.

‡ This line is not shown in Figs. 267, 268.

COROLLARY.—**If two circles touch one another the line joining their centres passes through the point of contact.**

In Figs. 269 and 270 the two circles may be regarded as intersecting at two points C, D, each of which coincides with the point P.

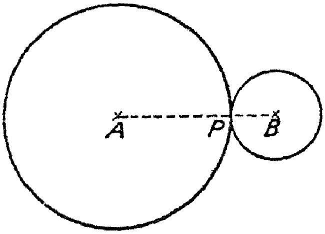

FIG. 269.

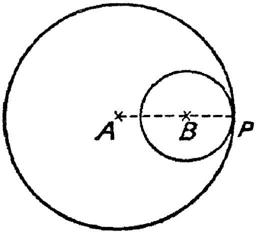

FIG. 270.

Now AB passes through the middle point of CD. Ch. 7.

But if C and D both coincide with P, the middle point of CD also coincides with P.

Hence AB passes through P.

181. *Prel. Ex.*—Two circles whose centres are A, B meet at a point P which does not lie in AB or AB produced. Prove that they also meet at one other point.

THEOREM [Tn.3].—**If two circles touch one another then** (i) **the join of their centres passes through the point of contact, and** (ii) **the distance between their centres is equal to the sum or difference of their radii.**

In Fig. 271 :—

Given that A, B are the centres of two circles x, y which touch at the point P (*i. e.* which meet at P and at no other point),

Required to prove that (i) P lies on AB or AB produced, and (ii) AB is equal to the sum or difference of the radii of x and y.

(i) If possible, let P *not* lie on AB or AB produced.

Cons.—Draw $PN \perp AB$, and produce it to Q making $NQ = PN$.

Proof.—$\because$ AB bisects PQ at right angles,
$\therefore$ $AQ=AP$ (L.3), *i. e.* Q lies on $\bigcirc^{ce}$ of $\odot$ x.
Similarly $BQ=BP$, *i. e.* Q lies on $\bigcirc^{ce}$ of $\odot$ y.

Hence $\odot$s x and y meet also at Q.

But they do not meet at any point other than P. *Hyp.*
Hence P must lie on AB or AB produced.

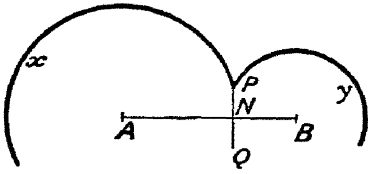

FIG. 271.

(ii) *Proof.*—CASE I.—If P lies on AB (Fig. 269),
Then AB is the sum of AP and PB,
that is AB is the sum of the radii of x and y.

CASE II.—If P lies on AB produced (Fig. 270),
Then AB is the difference of AP and BP,
that is AB is the difference of the radii of x and y.

COROLLARY.—**If two circles touch they have the same tangent at the point of contact.**

For in either Fig. 269 or Fig. 270 the line through P perpendicular to AB is the tangent to both circles.

182. **Theory of Two Circles.**—It is an interesting and valuable exercise to discuss the conditions which determine whether two circles cut, or touch, or do not meet. The following set of riders deals with the most important points:—

EXERCISES LXIII (Riders).

1. **If two circles meet at a point on the join of their centres, they do not meet at any other point. Also the distance between their centres is equal to the sum or difference of their radii.**

Take two cases, according as the point of meeting lies in the join or the join produced. In either case prove that the circles cannot meet at a second point by using Ch. 7.

2. If two circles meet at a point which is not on the join of their centres, they meet also at a second point. Also the distance between their centres is less than the sum and greater than the difference of their radii.

The first part of this theorem can be proved from the fact that both circles are symmetrical to the join of their centres; or it can be proved as in Tn. 3. To prove the second part use T. 4.

3. If two circles touch, the join of their centres passes through the point of contact.

This is Tn. 3. It follows from Ques. 2.

4. If the distance between the centres of two circles is greater than the sum of their radii each circle lies entirely without the other.

Prove by T. 4 that any point on the one circle lies outside the other.

5. If the distance between the centres of the two circles is less than the difference of their radii the smaller circle lies entirely within the larger.

Prove by T. 4 that any point on the smaller circle lies within the larger circle.

6. Find the locus of the centre of a circle of given radius which touches a given circle externally.

7. Find the locus of the centre of a circle of given radius which touches a given circle internally.

8. Find the locus of the centre of a circle which touches a given circle at a given point.

183. **Inscribed and Circumscribed Figures.**—If a circle lies within a rectilinear figure and touches all its sides, it is said to be **inscribed** in that rectilinear figure; and the latter is said to be **circumscribed** to, or **described** about the circle.

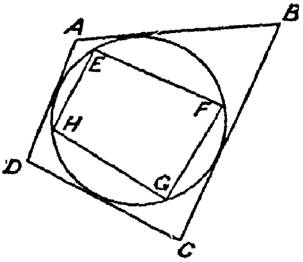

FIG. 272.

Thus in Fig. 272 the circle is inscribed in the quadrilateral *ABCD*, and the quadrilateral *ABCD* is circumscribed to the circle.

If a circle passes through all the vertices of a rectilinear figure it is said to be **circumscribed** to, or **described** about that rectilinear figure; and the latter is said to be **inscribed** in the circle.

Thus in Fig. 272 the circle is circumscribed to the quadrilateral *EFGH*, and the quadrilateral *EFGH* is inscribed in the circle.

EXERCISES LXIV.

RIDERS.

1. The line which joins the middle points of two parallel chords of a circle passes through the centre. Ch.1.

* Deduce by the method of limits that the line joining the points of contact of parallel tangents passes through the centre.

2. Show that all circles which have their centres on a given straight line, and meet at one fixed point not on this line, also meet one another at a second fixed point. Sy.2.

3. If a point lies within a circle, the longest straight line which can be drawn from that point to the circumference is the line which passes through the centre; and the shortest straight line is the line which will pass through the centre if produced. T.4.

4. If a point lies outside a circle, the longest straight line which can be drawn from that point to the circumference is the line which passes through the centre; and the shortest straight line is the line which will pass through the centre if produced. T.4.

5. OA is a fixed radius of a given circle ABC; show that any pair of tangents, which are drawn from the extremities of two radii equally inclined to OA, will intersect on OA produced. Tn.1; folding.

6. If two equal chords of a circle intersect, show that they divide each other into portions that are equal, each to each. Ch.1; Ch.4.

7. If two equal circles touch one another externally, every straight line drawn through their point of contact and terminated by the circumferences is bisected by the point of contact. C.2.

8. AA', BB', CC', are parallel chords of a circle; prove that the triangles ABC, $A'B'C'$ are equal. Ch.1; folding.

9. [Tn.4].—**If a quadrilateral is circumscribed to a circle then the sum of one pair of opposite sides is equal to the sum of the other pair.** Tn.2.

---

10. If two circles touch one another internally, and if any third circle be described touching both at different points show that the sum of the distance of the centre of the third circle from the centres of the two given circles is constant. Tn.3.

11. **If a circle touches the side BC of a triangle ABC at the point D, and touches the sides AB, AC *produced* at points E, F; prove that AE is half the perimeter of the triangle.** Tn.2.

12. If a quadrilateral is circumscribed to a circle prove that the sum of the two angles subtended at the centre by a pair of opposite sides is equal to two right angles. Tn.2.

13. Find the locus of the intersection of the two tangents at the extremities of a chord which is drawn at right angles to a fixed diameter. Folding.

14. Two given circles touch externally at *A*, and also touch a given line at points *P*, *Q*; show that *PAQ* is a right angle. Tn.2.

15. Find the greatest and least lines which have one extremity on each of two given circles, each of which lies outside the other. T.4.

16. Two circles touch the same line on the same side of it at two different points; show that the chords in one circle, which join the point of contact to the points where the circle meets the straight line joining the centres, are respectively parallel to the corresponding chords in the other circle. Tn.1; T.2.

17. Two equal circles intercept equal chords on a straight line; prove that the line is either parallel to or bisects the line joining the centres of the circles. Ch.4.

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18. Two circles, *ABC*, *ABD*, cut one another at *A* and *B*, and two parallel lines are drawn through *A* and *B*, terminated by the circumferences; prove that these lines are equal in length.

19. Find the locus of the centres of all circles which touch two given intersecting straight lines.

20*. Given that *AB*, *CD* are parallel chords of a circle, that *AC* and *BD* intersect at *P*, and that *AD* and *BC* intersect at *Q*, prove that the line *PQ* passes through the centre. Sy.3.

21*. If two circles touch one another internally at *A*, and a secant *PAQ* be drawn meeting the circumferences again at *P* and *Q*, prove that radii through *P* and *Q* will be parallel.

22*. If two circles touch one another externally at *A*, and a secant *PAQ* be drawn meeting the circumferences again at *P* and *Q*, prove that the radii through *P* and *Q* will be parallel.

23*. If two circles touch one another externally, the straight lines which join the opposite ends of parallel diameters will pass through the point of contact.

24*. If two circles touch one another internally, prove that the straight lines which join the extremities of parallel diameters towards the same parts will, when produced, pass through the point of contact.

25*. Find the locus of intersection of tangents to a circle which are inclined to one another at a fixed angle.

CALCULATIONS.

26. A circle of radius ·7″ rolls inside a fixed circle of radius 2·3″. What is the locus of the centre of the rolling circle?

27. A circle of radius ·4″ rolls outside a fixed circle of radius 1·8″. Find the locus of the centre of the rolling circle.

28. Three equal circles are drawn each touching the other two externally. Calculate the angle subtended at the centre of one circle by the arc joining the two points in which this circle touches the other circles.

29. A circle passes through the three angular points of an equilateral triangle. Calculate the angle subtended at the centre of the circle by a side of the triangle.

30. If in Fig. 264, $\angle LPM = 70^\circ$, find the magnitudes of $\angle$s LOP, OLM, LMP.

31. If in Fig. 264, $\angle LPM = 170^\circ$, find the magnitudes of $\angle$s LOM, OML, MLP.

32. If, in Fig. 264, $\angle LPM = x^\circ$, find the magnitudes of $\angle$s LOP, OLM, PLM.

33. Three circles of radii 2″, 3″, and 4″ respectively are drawn so that each touches the other two externally. Find the lengths of the sides of the triangle whose vertices are the centres of the circles.

34. Three circles of radii 2″, 3″, and 7″ respectively are drawn so that the two smaller circles touch externally and each touches the largest circle internally. Find the lengths of the sides of the triangle whose vertices are the centres of the circles.

35. A circle is inscribed in a triangle whose angles are 40°, 55°, 85°. Find the angles between the radii drawn to the points where the sides of the triangle touch the circle.

36. ABC is a circle, O its centre, and AC a diameter. If $\angle BOC = x^\circ$, find $\angle BAC$.

37. Find the length of a tangent drawn to a circle of radius 2″ from a point 5″ from the centre.

38. The tangent to a circle, drawn from a point 4″ from the centre, measures 3″. Find the length of the radius.

39. A chord of a circle is 2″ long and is at a distance 1″ from the centre of the circle. Find the length of the radius.

40. The radius of a circle is 10″. Find the distance from the centre of a chord whose length is 6″, and the length of a chord whose distance from the centre is 6″.

41. From a point A two tangents AB, AC are drawn to a circle whose centre is O. Given that the radius measures 1″, and that AO is 3″, find the lengths AB, AC, the area $OBAC$, and the length BC.

42. In a circle of radius 1″ there are two chords measuring 1·6″ and 1·2″. Which is nearer to the centre and by how much?

PROBLEMS IN PRACTICAL GEOMETRY.

43. Draw a circle of radius 1·5″. Take any point B one inch from the centre. Draw the chord whose middle point is B and measure its length.

44. A triangular plate of metal, each side of which measures 20 inches, can just be fitted into a circular hoop. Find, by a diagram drawn to scale, the radius of the hoop.

45. The sides of a triangle are 50″, 60″, 70″ respectively. Find, by a diagram drawn to scale, the radius of the circumscribed circle.

46. A chord of a circle is half the length of the radius; find the angle between the radii drawn to the extremities of the chord, when the radius of the circle is (i) 1″, (ii) 3 cms., (iii) 2·4″. What do you notice about the results? Can you give any explanation?

47. Draw a circle of radius 1 inch; draw any diameter AB, and on it mark off a series of points at distances ·2″, ·4″, ·6″, ·8″, 1″, 1·2″, 1·4″, 1·6″, 1·8″ respectively from A. Through each point draw a chord perpendicular to AB; bisect each half of each chord. Draw a curve (as simple a curve as possible) passing through all these bisecting points, and through A and B.

This curve is an "ellipse"; has this ellipse any axes of symmetry?

48. In the figure of the preceding question produce each half-chord beyond the circle so as to double its length. Draw as simple a curve as possible to pass through the extremities of these produced half-chords, and through A and B.

This curve is also an ellipse; has this ellipse any axes of symmetry?

49. In a circle of radius 15″ is a chord of length 20″. Find, by a diagram drawn to scale, the shortest distance from the middle point of the chord to the circumference.

50. AOB, COD are two perpendicular chords of a circle. Given that $AO = OB = 2$ cms., $CO = 1·5$ cms., construct the circle and measure OD.

51. Two parallel tangents to a circle are 2″ apart. Find the length of a parallel chord which is ·4″ from the nearer tangent.

52. Two parallel tangents to a circle are 4 cms. apart; a parallel chord measures 3 cms. Find the distance between the chord and the nearer tangent.

53. Two parallel chords of a circle measure 1″ each, and are 1·2″ apart. Find the radius of the circle.

CHAPTER XII.

THE CIRCLE: ANGULAR PROPERTIES.

184. DEFINITION.—**A *segment* of a circle is the figure contained by any arc of the circle and the chord which joins the extremities of that arc. This chord is the *base* of the segment.**

Thus, in Fig. 273 the chord *AB* divides the circle into two segments, viz. *ACB* and *ADB*, and *AB* is the base of either of these segments.

Any diameter divides a circle into two equal segments (Sy.2) each of which is a **semi-circle**; the arc of each semi-circle is half the circumference.

Any chord which is not a diameter divides a circle into two unequal segments. The segment which contains the centre is *greater* than a semi-circle and its arc is greater than half the circumference; the other segment is *less* than a semi-circle, and its arc is less than half the circumference (Fig. 273).

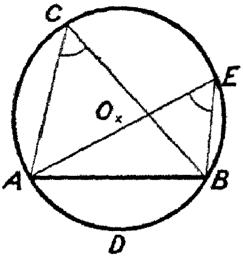

FIG. 273.

DEFINITION.—**The angle subtended by the base of a segment at any point on the arc of the segment is described as an angle "*in*" that segment.**

Thus, in Fig. 273 *ACB, AEB* are angles "in" the segment *ACB*.

DEFINITION.—**A *sector* of a circle is the figure bounded by any arc of the circle and the two radii drawn to the extremities of that arc.**

Thus, in Fig. 274 the two radii *OA,OB* divide the circle into two sectors; the smaller sector is bounded by the two radii and the arc *AHB*, and the larger sector by the two radii and the arc *ACB*.

The angle subtended by the arc of a sector at the centre is called **the angle** of the sector.

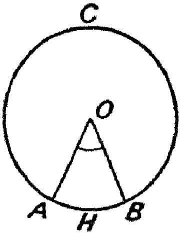

FIG. 274.

Thus, in Fig. 274 ∠*AOB* is the angle of the sector *OAHB*, and re-entrant ∠*AOB* is the angle of the sector *OACB*.

In Fig. 274 if $\angle AOB$ were a straight angle each of the two sectors would be a semi-circle. Thus a semi-circle is a sector whose angle is a straight angle.

Do not confuse *secant*, *segment*, *sector*. See Fig. 275.

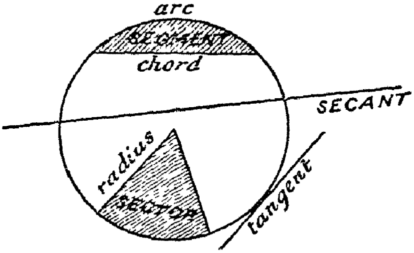

FIG. 275.

EXERCISES LXV.

1. Draw a semi-circle having a radius about 2″. Let AB be the diameter of the semi-circle, and take any points C, D, E, F on its circumference. Measure each of the angles ACB, ADB, AEB, AFB.

What do you infer about any angle in a semi-circle?

2. O is the centre of a circle, AB a diameter, and C any point on the $\bigcirc^{ce}$. Join OC, and then prove that $\angle ACB$ is a right angle. T.2; T.1.

3. Draw a circle of radius about 2″, and centre O; draw a chord AB of length about 3″, dividing the circle into two segments. Take three points C, D, E on the circumference of the larger segment, and three points F, G, H on the circumference of the smaller segment.

Measure and mark the size of the following angles:—AOB, ACB, ADB, AEB, AFB, AGB, AHB. Note which are acute and which obtuse.

What do you infer with regard to?—

(*a*) Angles in a segment greater than a semi-circle.

(*b*) Angles in a segment less than a semi-circle.

What relation appears to hold between?—

(*c*) Two angles in the same segment.

(*d*) Two angles which lie one in each of the two segments formed by the same chord.

(*e*) The angles subtended by the arc AGB at the centre and at some point on the remaining arc ADB.

(*f*) The angles subtended by the arc ADB at the centre and at some point on the remaining arc AGB.

Test your answers by drawing a second figure in which the radius of the circle is about 1·5″, and the chord AB about 2″ in length.

4. Lay a piece of paper on a smooth piece of wood,† and drive two pins A, B into the wood at a distance of about 2″ (Fig. 276). Place one of the set-squares PQR into several positions in succession, in each of which the two arms of the right angle R are in contact with the two pins; mark each position of R on the paper.

You thus obtain the locus of a point at which the line AB subtends a right angle. What is this locus?

[Compare the result with that of Question 1.]

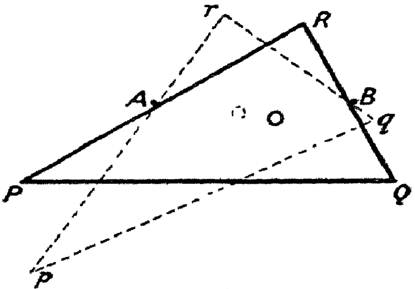

FIG. 276.

5. Repeat Question 4, using the set square whose angles are 30°, 60°, and 90°, and modifying the work so as to find (i) the locus of a point at which the line AB subtends an angle of 30°, and (ii) the locus of a point at which the *same* line AB subtends an angle of 60°.

What loci do you obtain? [Compare Question 3(*c*).]

(iii) Find O and P, the centres of each of part of locus (i), and measure $\angle AOB$ and $\angle APB$. Allowing for the fact that this work is not likely to be very accurate, what do you infer?

[Compare Question 3(*e*).]

6. Draw a circle of radius 2″ (Fig. 277). Draw the tangent at the lowest point A of the circle, also draw through A a chord of length 3″.

Measure the angle between chord and tangent (i) on the right of the chord, (ii) on the left of the chord. Measure any angle (i) in the segment on the left of the chord, (ii) in the segment on the right of the chord. Mark the magnitudes of these angles.

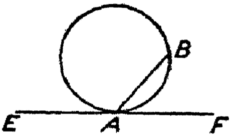

FIG. 277.

Repeat the process taking the radius of the circle 1·5″, and the chord 2″. What do you infer?

185. *Prel. Ex.*—(i) In Fig. 279 O is the centre of the circle. Prove that (*a*) $\angle BOE = 2\angle BAE$, (*b*) $\angle COE = 2\angle CAE$. What result do you obtain by adding?

(ii) In Fig. 281 O is the centre of the circle. Prove that (*a*) $\angle BOE = 2\angle BAE$, (*b*) $\angle COE = 2\angle CAE$. What result do you obtain by subtracting?

---

† With a transparent celluloid set square, the wood and pins are not necessary. It is sufficient to mark two points A, B on the paper, and to place the set square so that the two arms of the angle R pass through A and B.

THEOREM [AC.1].—**The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining arc of the circumference.**

In Figs. 278, 279, 280, 281 :—

Given that arc BDC of $\odot\ ABC$ subtends $\angle\alpha$ at the centre O, and subtends $\angle BAC$ at some point A on the arc BAC (the remaining arc of the circumference),

Required to prove that $\angle\alpha = 2\angle BAC$.

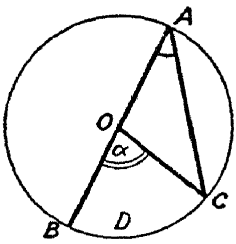

FIG. 278.

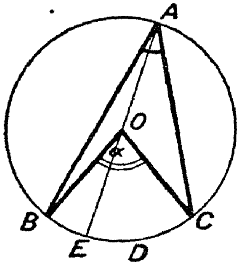

FIG. 279.

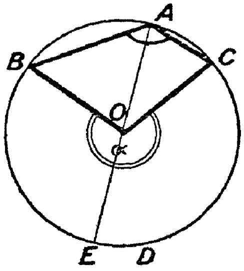

FIG. 280.

CASE I.—If the centre O lies on one arm, AB, of $\angle BAC$ (Fig. 278).

Proof.—In $\triangle OAC$, ext. $\angle\alpha = \angle A + \angle C$. T.1.

In $\triangle AOC$, $\because OA = OC$, $\therefore \angle C = \angle A$. T.2.

Hence $\angle\alpha = 2\angle A$.

CASE II.—If the centre O lies within $\angle BAC$ (Figs. 279, 280).†

Cons.—Join AO and produce it to meet $\bigcirc^{ce}$ again at E.

Proof.—$\because$ the centre O lies on the arm AE of $\angle EAC$,

$\therefore$, by Case I, $\angle EOC = 2\angle EAC$.

Similarly $\angle EOB = 2\angle EAB$. *Case* I.

Add the equals to the equals;

thus $\angle\alpha = 2\angle BAC$.

---

† It is obvious from Fig. 280 that when $\angle\alpha$ is a re-entrant angle or straight angle, the centre O must lie within $\angle BAC$.

CASE III.—If the centre O lies without $\angle BAC$ (Fig. 281).

Cons.—Join AO, and produce it to meet $\bigcirc^{ce}$ at E.

Proof.—$\because$ the centre O lies on the arm AE of $\angle EAC$.

$\therefore$, by Case I, $\angle EOC = 2\angle EAC$.

Similarly $\angle EOB = 2\angle EAB$. *Case* I.

Subtract the equals from the equals;

thus $\angle \alpha = 2\angle BAC$.

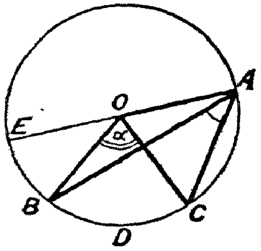

FIG. 281.

186. THEOREM [AC.2].—**Angles in the same segment of a circle are equal.**

In Figs. 282, 283, 284:—

Given that $\angle$s C, D are in the same segment $ACDB$,

Required to prove that $\angle C = \angle D$.

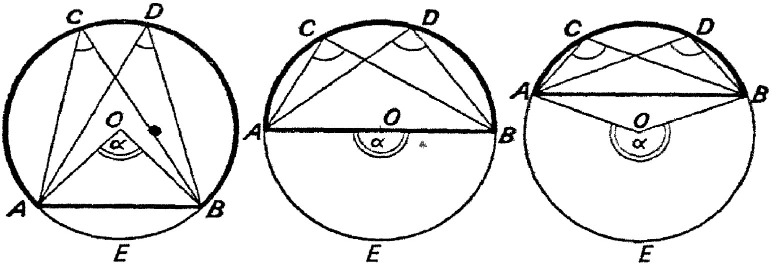

FIG. 282. FIG. 283. FIG. 284.

CASE I.—When segment $ACDB >$ semi-circle (Fig. 282).

CASE II.—When segment $ACDB$ is a semi-circle (Fig. 283).

CASE III.—When segment $ACDB <$ semi-circle (Fig. 284).

Cons.—Let AEB be the arc which completes the circle, and O the centre of the circle. Join OA, OB.

Proof.—In all three cases:—

$\because$ arc AEB subtends $\angle\alpha$ at the centre O, and $\angle$s C and D at points on the remainder of $\bigcirc^{ce}$,

$\therefore$ $\angle C = \frac{1}{2}\angle\alpha$; and $\angle D = \frac{1}{2}\angle\alpha$. AC.1.

Hence $\angle C = \angle D$.

187. DEFINITION.—**Four points which lie on the same circle are called** *concyclic points*. **If the four vertices of a quadrilateral lie on the same circle it is called a** *cyclic quadrilateral*.

Prel. Ex.—Show that it is not possible to draw two different segments of circles on the same side of the same base to contain equal angles.

THEOREM [AC.2c].—**If the line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.**

In Fig. 285:—

Given that $\angle ACB = \angle ADB$,

Required to prove that the points A, B, C, D, are concyclic.

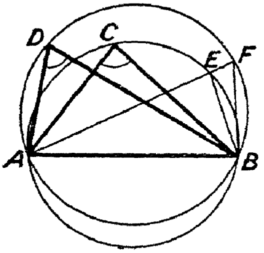

FIG. 285.

Cons.—Let a circle be drawn through A, B, C, and a second circle through A, B, D.

If possible suppose that these two circles do not coincide.

Through A draw a straight line AF lying within each of the angles CAB, DAB. Then AF will meet the two circles ACB, ADB at some points E, F.†

Proof.—In segment ACB, $\angle AEB = \angle C$, AC.2.
and in segment ADB, $\angle AFB = \angle D$. AC.2.

Also $\angle C = \angle D$; hence $\angle AEB = \angle AFB$.

But this is impossible;
for in $\triangle EFB$, ext. $\angle AEB > \angle AFB$, T.1.
(or if E lies in AF produced, ext. $\angle AFB > \angle AEB$). T.1.

Hence the two circles ACB, ADB must coincide, **that is** the four points A, B, C, D, are concyclic.

---

† The proof of this proposition usually given in text-books is defective; for it assumes that either AD or BD necessarily meets the arc ACB. It is easy to modify the usual proof so as to discuss all possible cases, including the special case where ACD is a straight line; but the proof here given is logically complete and more concise.

COROLLARY [L. 6].—**The locus of the vertex of a triangle drawn on one side of a given base and having a given vertical angle is the arc of a segment of a circle on that base.**

For if AB is the given base (Fig. 286), and C, D, E, F . . . different positions of the vertex, then by hypothesis

$$\angle ACB = \angle ADB = \angle AEB = \angle AFB = \&c.$$

Hence $\odot$ ABC passes through $D, E, F,$ . . . and *all* other possible positions of the vertex on the same side of AB. *Proved.*

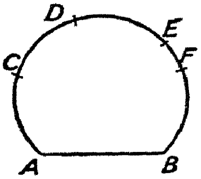

FIG. 286.

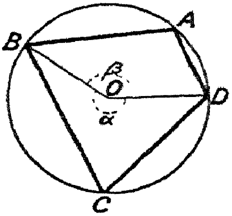

FIG. 287.

188. THEOREM [AC. 3].—**If a quadrilateral be inscribed in a circle the sum of each pair of opposite angles is equal to two right angles.**

Given that $ABCD$ is a cyclic quadrilateral (Fig. 287), *Required to prove* that $\angle A + \angle C = \angle B + \angle D = 2R$.

Cons.—Let O be the centre of the circle. Join OB, OD.

Proof.—$\because$ arc BCD subtends $\angle\alpha$ at centre O, and $\angle A$ at a point on the remaining arc of the $\bigcirc^{ce}$,

$$\therefore \angle A = \tfrac{1}{2} \angle \alpha. \qquad \text{AC.1.}$$

$\because$ arc BAD subtends $\angle\beta$ at centre O, and $\angle C$ at a point on the remaining arc of the $\bigcirc^{ce}$,

$$\therefore \angle C = \tfrac{1}{2} \angle \beta. \qquad \text{AC.1.}$$

Hence $\angle A + \angle C = \tfrac{1}{2}\angle\alpha + \tfrac{1}{2}\angle\beta$.

But $\angle\alpha + \angle\beta = 4R$; A.1.

hence $\angle A + \angle C = 2R$.

Similarly by joining OA, OC it can be proved that

$$\angle B + \angle D = 2R.$$

Corollary 1.—**In a cyclic quadrilateral any exterior angle is equal to the opposite interior angle.**

In Fig. 288, $\because$ *CDE* is a st. line, $\therefore$ $\angle\theta + \angle\lambda = 2\text{R}$. A.1.
Also $\angle B + \angle\lambda = 2\text{R}$. *Proved.*
Hence $\angle\theta + \angle\lambda = \angle B + \angle\lambda$;
therefore $\angle\theta = \angle B$.

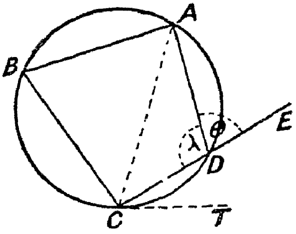

Fig. 288.

Corollary 2.*—**In Fig. 288, if *CT* is the tangent at *C*, then $\angle ACT = \angle B$.**

For if *D* be supposed to travel along the curve toward *C*, then (in *all* positions of *D*) $\angle ADE = \angle B$.

But when *D* moves up to *C*, *AD* becomes *AC*, and secant *CDE* becomes tangent *CT*. Thus $\angle ADE$ becomes $\angle ACT$.

Hence $\angle ACT = \angle B$.

189. Theorem [AC.3c].—**If two opposite angles of a quadrilateral are together equal to two right angles then the vertices of the quadrilateral are concyclic.**

In the quadrilateral *ABCD* (Fig. 289) :—

Given that $\angle A + \angle C = 2\text{R}$,

Required to prove that the points *A*, *B*, *C*, *D* are concyclic.

Cons.—Describe a circle to pass through the points *B*, *C*, *D*.

Take any point *X* on the ○$^{\text{ce}}$ of this ⊙, and on the same side of *BD* as *A*. Join *BX*, *DX*.†

---

† The usual proof of this proposition is defective, as it assumes that *BA* necessarily meets the circle *BCD* on the upper side of *BD*—an assumption which fails if *A* lies a long way above *BD*.

Proof.—$\because$ *BCDX* is a quad. inscribed in a $\odot$, Cons.

$\therefore \angle X + \angle C = 2R$. AC.3.

But $\angle A + \angle C = 2R$;

$\therefore \angle X + \angle C = \angle A + \angle C$; whence $\angle X = \angle A$.

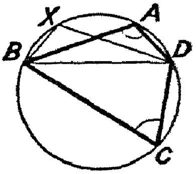

Fig. 289.

$\because$ *BD* subtends equal angles at *X* and *A*,

$\therefore$ *A* lies on $\odot$ *BXD*. AC.2r.

That is *A* lies on $\odot$ *BCD*.

EXERCISES LXVI (*Riders*).

1. *AB* and *CD* are two chords of a circle intersecting at *E*; prove that the triangles *AEC* and *BED* are equiangular.

2. *ABC* is an acute-angled triangle, and perpendiculars *AD, BE, CF* are drawn to the three sides; prove that the following sets of points are concyclic:—(1) *B, F, E, C*; (2) *A, F, D, C*; (3) *A, E, D, B*.

3. *ABC* is an acute-angled triangle, and perpendiculars *AD, BE, CF* drawn to the three sides, are found to meet at one point *P*; prove that the following sets of points are concyclic:—(1) *P, D, C, E*; (2) *P, E, A, F*; (3) *P, F, B, D*.

4. If a triangle be inscribed in a circle, prove that the sum of the angles in the three segments exterior to the triangle is equal to four right angles. AC.3.

5. If *AB, CD,* two chords in a circle, intersect in *E*, prove that the angle *AED* is equal to half the sum of the angles *AOD, BOC, O* being the centre of the circle. AC.1.

6. Given the base of a triangle in magnitude and position, and the sum of the base angles, what is the locus of the vertex? L.5.

190. *Prel. Ex.*—(i) What angle is subtended at the centre of a circle by the whole circumference of the circle?

(ii) What kind of angle is subtended at the centre of a circle by (*a*) an arc equal to half the $\bigcirc^{ce}$, (*b*) an arc less than half the $\bigcirc^{ce}$, (*c*) an arc greater than half the $\bigcirc^{ce}$?

(iii) What angle is subtended at a point on the remaining arc by (*a*) an arc equal to half the $\bigcirc^{ce}$, (*b*) an arc less than half the $\bigcirc^{ce}$, (*c*) an arc greater than half the $\bigcirc^{ce}$?

THEOREM [AC.4].—(i) **Any angle in a semi-circle is a right angle.** (ii) **Any angle in a segment greater than a semi-circle is less than a right angle.** (iii) **Any angle in a segment less than a semi-circle is greater than a right angle.**

In Figs. 290, 291, 292, $\angle C$ is an angle in the segment ACB:—

Given that segment ACB is (i) a semi-circle (Fig. 290), (ii) greater than a semi-circle (Fig. 291), (iii) less than a semi-circle (Fig. 292),

Required to prove that

(i) $\angle C = \text{R}$, (ii) $\angle C < \text{R}$, (iii) $\angle C > \text{R}$.

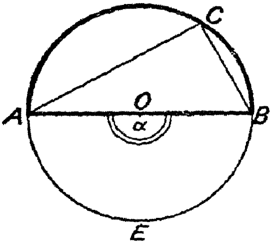

FIG. 290.

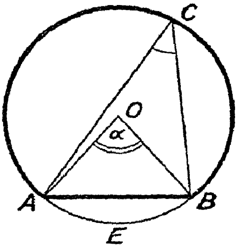

FIG. 291.

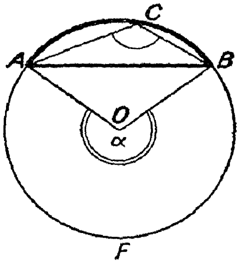

FIG. 292.

Cons.—Let AEB be the arc which completes the $\bigcirc^{ce}$ and O the centre of the circle. Join OA, OB.

Proof.—In all three figures, arc AEB subtends $\angle \alpha$ at the centre and $\angle C$ at a point on the remaining arc;

hence in all three figures $\angle C = \frac{1}{2} \angle \alpha$ AC.1.

(i) In Fig. 290, $\angle \alpha = 2\text{R}$,

Hence $\angle C = \text{R}$.

(ii) In Fig. 291, $\angle \alpha < 2\text{R}$,

Hence $\angle C < \text{R}$.

(iii) In Fig. 292, $\angle \alpha > 2\text{R}$,

Hence $\angle C > \text{R}$.

COROLLARY.—**If a circle be circumscribed to a right-angled triangle its centre will lie on the hypotenuse.**

For otherwise the angle opposite to the hypotenuse will be an angle in a segment of a circle which is either greater than or less than a semi-circle; and in either case this angle cannot be a right angle.

191. *Prel. Ex.*—AC is a diameter of a circle. The tangent AF, and a chord AB are drawn on the same side of AC. Prove that $\angle BAF = \angle BCA$.

THEOREM [AC.5].—**If a chord be drawn from the point of contact of a tangent to a circle, then each of the angles which the chord makes with the tangent is equal to the angle in the alternate segment of the circle.**

In Fig. 293:—

Given that EAF is a tangent to the ⊙; and AB a chord,

Required to prove that

(i) $\angle BAF$ = angle in segment BCA,

and (ii) $\angle BAE$ = angle in segment BDA.†

Cons.—Through A draw the diameter AC.

Take any point D on the arc ADB.

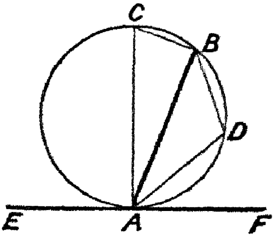

FIG. 293.

Join AD, BD, BC.

Proof.—(i) In semi-circle CDA, $\angle CBA = R$. AC.4.

Hence, in $\triangle ABC$, $\angle BAC + \angle BCA = R$. RT.1.

$\because CA$ is a diameter and EF a tangent

$\therefore \angle CAF = R$. Tn.1.

Hence $\angle CAF = \angle BAC + \angle BCA$.

From these equals subtract $\angle BAC$;

thus $\angle BAF = \angle BCA$.

Hence BAF = any angle in segment BCA. AC.2.

(ii) $\because EAF$ is a st. line, $\therefore \angle BAF + \angle BAE = 2R$. A.1.

$\because \angle ADBC$ is a cyclic quad., $\therefore \angle C + \angle D = 2R$. AC.3.

Hence $\angle BAF + \angle BAE = \angle C + \angle D$.

But $\angle BAF = \angle C$; *Proved.*

take the equals from the equals, thus $\angle BAE = \angle D$.

Hence $\angle BAE$ = any angle in segment BDA. AC.2.

---

† That is, angle on *right* of AB = angle in segment on *left* of AB, and angle on *left* of AB = angle in segment on *right* of AB. Hence the use of the word "alternate" in stating the proposition.

192. THEOREM [AC.5c].—**If a line drawn through one extremity of a chord of a circle makes an angle on one side of the chord equal to the angle in the alternate segment, then this line is a tangent to the circle.**

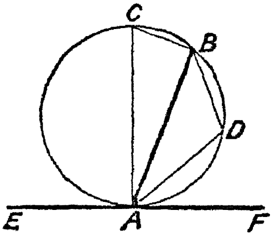

FIG. 293.

In Fig. 293 :—

Given that $\angle EAB$ = angle in segment BDA,

Required to prove that EAF is the tangent at A.

Cons.—Let AX be the tangent at A on the same side of AB as AE. (AX is not shown in the figure.)

Proof.—$\because$ AX is a tangent (*Cons.*), and AB a chord,

$\therefore \angle BAX$ = angle in alt. segment BDA. AC.5.

Also $\angle BAE$ = angle in alt. segment BDA. *Hyp.*

Hence $\angle BAX = \angle BAE$, *i.e.* AX coincides with AE.

Therefore EAF is the tangent at A.

EXERCISES LXVII (Riders).

1. Two circles intersect at P, Q; PA, PB are diameters of the two circles. Prove that the straight line AB passes through Q.

2. If two circles are described having the equal sides of an isosceles triangle as diameters, they intersect at the middle point of the base.

3. In any triangle, the two circles described on two sides as diameters intersect on the third side.

4. ABC is a triangle. Show how to draw the tangent at A to the circumscribed circle, without finding the centre of this circle. AC.5c.

193. THEOREM [AC.6].—**In equal circles if two arcs subtend equal angles at the centre they are equal.**

In Fig. 294 O and P are the centres of the two equal circles ABC, DEF:—

Given that $\angle O = \angle P$,

Required to prove that arc AHB = arc DKE.

Cons.—Suppose $\odot DEF$ to be applied to $\odot ABC$ in such a way that P falls on O, PD lies along OA, and E and B fall on the same side of OA.

Proof.—$\because PD = OA$ (*Hyp.*), $\therefore D$ falls on A;
$\because \angle DPE = \angle AOB$, $\therefore PE$ lies along OB;
$\because PE = OB$ (*Hyp.*), $\therefore E$ falls on B.

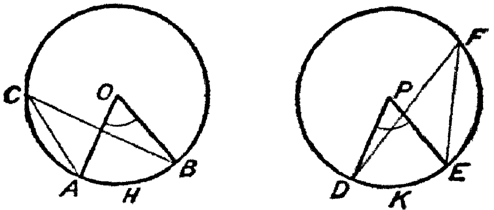

FIG 294.

Again, because the two circles have the same centre and equal radii,

therefore their circumferences coincide.

Hence arc DKE coincides with arc AHB;

that is arc $DKE =$ arc AHB.

COROLLARY 1.—**In equal circles if two arcs subtend equal angles at points on the remaining arcs of the circumferences they are equal.**

For if, in Fig 294, $\angle C = \angle F$;
then the doubles of these angles are equal,
i. e. $\angle O = \angle P$ AC.1.

Hence arc $AHB =$ arc DKE;

COROLLARY 2.—**In equal circles equal chords cut off equal arcs.**

For, in Fig. 294, if chord $AB =$ chord DE,
then $\angle O = \angle P$. Ch.5.

Hence arc $AHB =$ arc DKE;

that is the two smaller arcs cut off by the chords AB, DE are equal.

Also the two circumferences are equal;

hence remaining arc $ACB =$ remaining arc DFE;

that is the two larger arcs cut off by the chords AB, DE are equal.

COROLLARY 3.—**In the same circle if two arcs subtend equal angles at the centre or at points on the circumference they are equal; also equal chords cut off equal arcs.**

194. THEOREM [AC.6c].—**In equal circles, if two arcs are equal they subtend equal angles at the centres.**

In Fig. 294, O and P are the centres of the two equal circles ABC, DEF:—

Given that arc $AHB =$ arc DKE,

Required to prove that $\angle O = \angle P$.

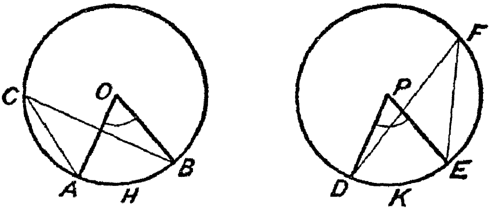

FIG. 294.

Cons.—Apply $\odot DEF$ to $\odot ABC$, as in the preceding theorem, so that PD coincides with OA, and $\bigcirc^{ce}$ of $\odot DEF$ coincides with $\bigcirc^{ce}$ of $\odot ABC$.

Proof.—$\because$ arc DKE lies along arc AHB, and also arc $DKE =$ arc AHB, $\therefore E$ falls on B.

But P falls on O; $\therefore PE$ falls on OB.

Hence $\angle P$ coincides with $\angle O$, that is $\angle P = \angle O$.

COROLLARY 1.—**In equal circles if two arcs are equal they subtend equal angles at points on the remaining arcs of the circumferences.**

For the angles which they subtend at points on the circumferences are halves of the angles which they subtend at the centres. AC.1.

COROLLARY 2.—**In equal circles equal arcs are cut off by equal chords.**

For if (Fig. 294) arc $AHB =$ arc DKE, then $\angle O = \angle P$.

Hence chord $AB =$ chord DE. Ch.5.

COROLLARY 3.—**In the same circle if two arcs are equal they subtend equal angles at the centre, or at points on the remaining arc of the circumference, and are cut off by equal chords.**

NOTE.—In the two theorems AC.6 and AC.6c the argument applies equally well to the arcs ACB, DFE and the re-entrant angles O and P.

EXERCISES LXVIII.

CALCULATIONS.

1. An arc of a circle subtends a right angle at the centre. What fraction is this arc of the whole circumference?

2. An arc of a circle subtends an angle of 60° at a point on the remaining arc of the circle. What fraction is this arc of the whole circumference?

3. If the circumference of a circle is divided into nine equal arcs, what angle does each of these arcs subtend at the centre, and what angle at a point on one of the other arcs?

4. An arc of a circle subtends an angle of 100° at the centre. What angle does the remaining arc of the circle subtend at the centre, and what angle at some point on the first arc?

5. The chord of a given arc is equal to the radius of the circle. What fraction is this arc of the whole circumference?

6. AB is a chord of a circle and AC a tangent at the point A. If $\angle BAC = x^\circ$, find the angle which AB subtends at the centre of the circle, and also at a point on the arc of the larger segment:—
(i) if $\angle BAC$ is acute, (ii) if $\angle BAC$ is obtuse.

RIDERS.

7. In equal circles, sectors are equal if their angles at the centres are equal.

8. If a set of radii make equal angles with each other, they divide the circumference into equal arcs.

9. AB and CD are two parallel chords in a circle; prove that the arc AC is equal to the arc BD. AC. 6.

10. The straight lines which join the extremities of two equal arcs in a circle are either parallel or equal. AC. 6c.

11. The straight lines which join the extremities of parallel chords in a circle are equal. AC. 1, Ch. 5.

12. BAC is a given segment of a circle, and BAC is any angle in the segment; prove that the straight line which bisects the angle BAC always passes through a fixed point on the circumference. AC. 6.

195. **On the Circumference of a Circle.**—The object of the following exercises is to investigate the relation between the circumference of a circle and its diameter. We shall find that the result obtained by dividing the circumference by the diameter *is the same for all circles.* At present the student can only show this by actual measurement; the relation can also be proved by mathematical reasoning, but this is a difficult process and must be deferred.

EXERCISES LXIX.

[*These exercises should be worked with great accuracy.*]

1. Measure the circumference and diameter of (i) a penny, (ii) a halfpenny, and in each case work out the quotient

circumference ÷ diameter.

[First rule a straight line on paper. Mark the coin with a scratch at the edge of either face. Stand the coin upright on the line with the scratch against the paper and mark the position of the scratch on the paper. Then carefully roll the coin along the line, *without letting it slip,* and mark the point where the scratch again comes in contact with the paper. Measure the distance between these two marks, which is equal to the circumference of the coin.

Repeat this two or three times, and take the average of your results as the correct value of the circumference.

Then measure the diameter with the scale; and find the required quotient.]

2. Measure the circumference and diameter of the flat cylinder contained in your box of models of geometrical solids, and work out the quotient

circumference ÷ diameter.

[To measure the circumference, wrap a strip of paper tightly round the cylinder so that the two ends of the paper overlap. Using a pin, prick once through the two folds of the paper at the overlapping. Spread the strip of paper flat, and measure *the distance between the two pricks.*]

3. If an arc of a circle subtends an angle of 30° at the centre, what fraction is this arc of the whole circumference?

4. Draw a sector of a circle making the angle of the sector exactly† 30° and the radius exactly 3·5″; let O be the centre and AB the arc of the sector (Fig. 295).

On the arc of this sector mark distinctly two points H, K, dividing the arc into three portions which are *roughly* equal. Measure the straight lines AH, HK, KB as accurately as possible, and add the results.

The sum is very nearly the exact length of the arc AB (for the difference between the lengths of the chord AH and the arc AH is exceedingly small, and so on).

Multiply this result by 12; the product is very nearly the exact length of the circumference of the whole circle (see Question 3).

Give as accurately as you can the lengths of the diameter and circumference of this circle.

---

† Probably this angle can be constructed more accurately *without* using the protractor.

5. Work again through Question 4 using a radius of 3·5 cms. instead of 3·5″. Give as accurately as you can the lengths of the diameter and circumference of this circle.

6. The ratio of the circumference of a circle to its diameter (*i. e.* the number of times which the diameter is contained in the circumference) can be expressed very accurately as a vulgar fraction whose denominator is 7. What fraction do you suggest for this ratio?

Compare the results of Questions 4 and 5.

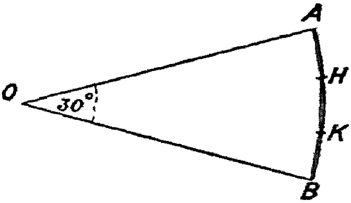

FIG. 295.

From the preceding Exercises it may be inferred that the ratio of the circumference of a circle to its diameter is **constant**, *i. e.* is the same for all circles. The student must commit to memory the value of this ratio, both in its fractional and decimal forms, viz. $\frac{22}{7}$ and 3·14. Neither of these values is exactly correct; the first is too large, and the second too small; in each case the error is about 1 in 2000.

The exact value of this ratio is usually denoted in mathematics by the Greek letter π. Thus we have, in any circle

$$\frac{\text{circumference}}{\text{diameter}}=\pi;\quad i.\,e.\ \textit{circumference}=\pi\times\textit{diameter};$$

$$\text{where}\qquad \pi=\tfrac{22}{7}\ \text{(approx.)}=3{\cdot}14\ \text{(approx.)}$$

As this result has been obtained by experiment, and not by mathematical reasoning, it must not be admitted as part of the logical course of geometrical argument, and cannot be used in the proof of any proposition or rider. It will, however, be often used in the remainder of the book in calculations.

196. **Calculations.**—The following example is a good instance of a problem which requires both calculation and accurate drawing:—

Example.—*Find the length of the arc cut off from a circle of radius* 100 *yards by a chord of length* 80 *yards.*

A convenient scale of representation is 50 yards to the inch. The radius is then represented by a length of 2″, and the chord by 1·6″.

With centre *O*, draw ⊙ *ABC* of radius 2″ (Fig. 296). In it draw a chord *AB* of length 1·6″.

To determine the length of the arc AB, **measure** *the angle which it subtends at the centre, and thence* **calculate** *its length.*

Draw ∠ *AOB*. By measurement we find ∠ $AOB = 47 \cdot 2°$.

Since arc *AB* subtends ∠ 47·2° at centre, and the whole circumference subtends ∠ 360° at centre,

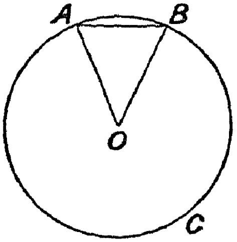

FIG. 296.

$$\therefore \text{ arc } AB = \frac{47 \cdot 2}{360} \times \text{circumference}.$$

$$\text{Now circumference} = \frac{22}{7} \times \text{diam.} = \frac{22}{7} \times 4''.$$

$$\text{Hence} \qquad \text{arc } AB = \frac{47 \cdot 2}{360} \times \frac{22}{7} \times 4''.$$

But 1″ represents 50 yards.

Hence arc *AB* represents a length of

$$\frac{47 \cdot 2}{360} \times \frac{22}{7} \times 4 \times 50 \text{ yards}.$$

Evaluating this we obtain **82·4 yards.**

NOTE.—Since ∠ *AOB* measures 47·2°, ∴ the corresponding angle on the original figure also measures 47·2°. Hence we could calculate that required arc $= \frac{47 \cdot 2}{360} \times$ circumference of original circle

$$= \frac{47 \cdot 2}{360} \times \left(\frac{22}{7} \times 200\right) \text{ yards} = 82 \cdot 4 \text{ yards}.$$

EXERCISES LXX.

CALCULATIONS.

1. An arc of a circle subtends $x°$ at the centre; what fraction is this arc of the whole circumference?

2. An arc of a circle subtends $x°$ at a point on the remaining arc; what fraction is this arc of the whole circumference?

3. A chord of a circle is equal to the radius. If the radius of the circle measures 7″, find the lengths of the arcs into which this chord divides the circumference.

4. What is the circumference of a circle whose radius is 10″, and what is the radius of a circle whose circumference is 10″?

5. An arc of a circle is equal to the radius of the circle. (i) What fraction is this arc of the whole circumference? (ii) What angle does this arc subtend at the centre of the circle? (iii) What angle does this arc subtend at a point on the remaining arc?

6. The diameter of a bicycle wheel is 28″. Find its circumference, and determine how many turns it makes in running a mile.

7. The piston of a locomotive makes 240 double strokes in running a mile; find the diameter of the driving wheel.

8. A chord of a circle is inclined to the tangent at one extremity at an angle of 40°. If the radius of the circle is 3″, find the length of the smaller arc formed by this chord.

9. AB, AC are tangents to a circle from the point A; B and C are the points of contact. If $\angle A = 100^\circ$, determine the angle in the smaller segment formed by the chord BC.

10. From a point A a tangent AB is drawn to a given circle meeting it at B, and a secant ACD meeting the circle at C and D. Given that $\angle ABC = 40^\circ$, $\angle CBD = 70^\circ$, find $\angle BAC$.

11. ABC is a triangle inscribed in a circle whose centre is O. Given that $\angle A = 60^\circ$, $\angle B = 40^\circ$, $\angle C = 80^\circ$, find $\angle OAB$ and $\angle OAC$.

12. ABC is a triangle inscribed in a circle whose centre is O. Given that $\angle A = 110^\circ$, $\angle B = 30^\circ$, $\angle C = 40^\circ$, find $\angle OBC$ and $\angle OBA$.

13. Two circles PQR, PQS intersect at the two points P, Q. The common chord PQ lies between the centres of the circles and subtends an angle of 80° at the centre of the first circle and 40° at the centre of the second. A straight line RPS, passing through P, meets the circumferences again at R and S. Find $\angle RQS$.

14. A triangle is inscribed in a circle. If the angles of the triangle are 40°, 60°, 80°, and the circumference measures 3 ft., find the lengths of the arcs into which the vertices divide the circumference.

15. ABC is a triangle inscribed in a circle; and the tangents at A, B, and C form a triangle PQR. If the angles of the triangle ABC measure 50°, 60°, 70° respectively, find the angles of the triangle PQR.

16. $HKLM$ is a cyclic quadrilateral whose diagonals intersect at O. If $\angle MHL = 30^\circ$, $\angle KHL = 40^\circ$, $\angle KLH = 50^\circ$, find $\angle HML$ and $\angle HOK$.

BY CONSTRUCTION AND CALCULATION.

17. In $\triangle ABC$, $AB = 1''$, $BC = 2''$, $\angle B = 90^\circ$. Bisect BC at D. In AD produced find a point E concyclic with A, B, C. Measure DE.

18. Draw a square $ABCD$ of side 2″. Bisect BC at E; find a point F in DC which is concyclic with A, D and E. Measure DF.

19. In $\triangle HKL$, $HK = 2''$, $HL = 2''$, $\angle H = 120^\circ$. Bisect HL at M. Find a point N in LK, concyclic with H, K, M. Measure KN.

20. Find the length of the minor arc cut off from a circle of radius 2″ by a chord of length 3″.

21. Construct a circle whose circumference measures 10″. Find the distance between the extremities of an arc which measures 2″.

22. The radius of a circle is 10 feet, and a chord measures 13 feet. Find the length of the smaller arc which stands on this chord.

23. The circumference of a circle measures 12 feet. Find the length of a chord which cuts off a segment containing an angle of 50°.

24. The sides of a triangle measure 5 cms., 5 cms., 6 cms. If a circle is circumscribed to this triangle, find the lengths of the arcs into which the circumference is divided by the vertices of the triangle.

197. Revise § 144. The following example affords a good illustration of the method of solving a rider:—

Example.—*$ABCD$ is a cyclic quadrilateral; E, F, G, H are the middle points of the arcs AB, BC, CD, DA respectively. Prove $EG \perp FH$.*

(In the following discussion of the solution the reference numbers refer to the hints given in § 144.)

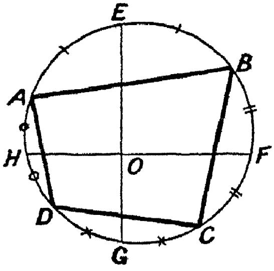

Fig. 297.

Analysis.—(1) Draw the diagram as in Fig. 297. The four sides of the quadrilateral are purposely made *unequal*. The pairs of equal arcs are indicated by strokes, crosses, etc.

Required to prove that $\angle EOF = R$.

(2) The *hypothesis* states (*a*) that $ABCD$ is a cyclic quadrilateral and (*b*) that four arcs are bisected; (*a*) suggests the use of Theorem AC.3. The *conclusion*, that $\angle EOF = R$, suggests that we should try to prove $\angle EOF = \angle EOH$, or $\angle EOF = \angle FOG$, by means of congruent triangles; but there do *not* appear to be any congruent triangles available.

(3) In the cyclic quad., $\angle A + \angle C = \angle B + \angle D = 2R$. No other facts appear to be suggested either by the hypothesis or by the conclusion.

(4) It is obvious that the figure in its present condition does not give sufficient facts to prove the rider; it is therefore advisable to introduce some new lines. Such lines should be lightly drawn in pencil and only filled in with ink when it is obvious that they will be required.

In drawing new lines, it is necessary to bear in mind—(i) that it is essential to bring the *bisected arcs* into the argument (see the 5th hint in § 144); (ii) that some method has to be devised for proving $\angle EOF = R$, and that congruent triangles are apparently not available.

Join EF (Fig. 298).

To prove that $\angle EOF = R$ requires that $\angle E + \angle F = R$. T.1.

Now $\angle E$ stands on arc FG; and obviously arc $FG = \frac{1}{2}$ arc BCD.

But $\angle A$ stands on arc *BCD*.
Hence, by an easy deduction from AC. 6*c*, *Cor.* 3, $\angle E = \frac{1}{2}\angle A$.
Similarly $\angle F$ stands on arc *EH*, and $\angle C$ stands on arc *BAD*;
thus $\angle F = \frac{1}{2}\angle C$.
Thus $\angle E + \angle F = \frac{1}{2}\angle A + \frac{1}{2}\angle C = \text{R}$.

The solution is now obvious, and may be set out as follows:—

Cons.—Join *EF*.

Proof.—$\because$ arc $CF = \frac{1}{2}$ arc *CB*, and arc $CG = \frac{1}{2}$ arc *CD*, $\therefore$ arc $FG = \frac{1}{2}$ arc *BCD*.

But $\angle E$ stands on arc *FG*, and $\angle A$ stands on arc *BCD*, $\therefore$ $\angle E = \frac{1}{2}\angle A$. AC. 6*c*.
Similarly $\angle F = \frac{1}{2}\angle C$. AC. 6*c*.
Hence $\angle E + \angle F = \frac{1}{2}\angle A + \frac{1}{2}\angle C$.

In cyclic quad. *ABCD*, $\angle A + \angle C = 2\text{R}$. (AC. 3). **Hence**, $\angle E + \angle F = \text{R}$.

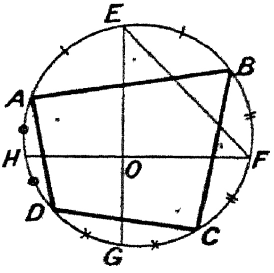

FIG. 298.

198. **On Constant Quantities.**—It frequently happens that if a geometrical figure is supposed to alter in accordance with certain given conditions, then some magnitude connected with the figure will be found to remain **constant**, *i. e.* will be the same for all possible cases of the figure.

For example, if *XY* (Fig. 299) is a given line and *ABC* a given circle whose centre is *X*, and if a triangle is to be drawn on base *XY* and having its vertex *Z* on the circumference of the circle; then the triangle will vary according to the position of *Z* on the circumference, but *the side XZ will be constant*; for all radii of the circle have the same length.

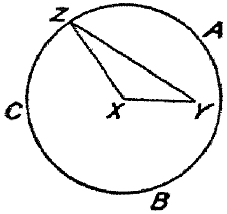

FIG. 299.

Again, if *ACB* is a segment of a circle on base *AB* (Fig. 300), and if a triangle is drawn having *AB* for base and having its vertex *V* on the arc *ACB*, then the triangle will vary according to the position of *V* on the arc, but its *vertical angle V will be constant;* for all angles in the segment *ACB* are equal.

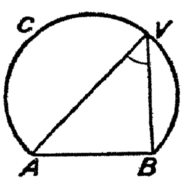

FIG. 300.

For other illustrations of constant quantities see § 143.

199. The following examples will serve as models for the solution of riders involving constant quantities :—

Example 1.—*A point P moves along the base BC of an isosceles triangle ABC. Prove that the sum of the distances of P from the sides AB, AC is constant.*

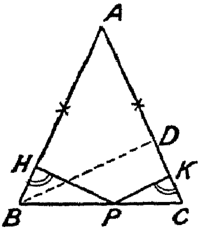

FIG. 301.

Analysis.—Draw the diagram (Fig. 301).

Given that $AB = AC$, and that PH, PK are $\perp$ AB, AC respectively,

Required to prove that, as P moves along BC, $PH + PK$ is constant.

P may lie at any point on BC; suppose that P is moved to B; then the distance of P from AB becomes zero, and the distance of P from AC becomes BD, where $BD \perp AC$.

Thus, if the sum of these distances does not alter as P moves, then $PH + PK$ must be equal to $0 + BD$, *i. e.* to BD.

Conversely, to prove that $PH + PK$ is constant, **it will be sufficient to prove that** $PH + PK = BD$.

If PL is drawn $\perp$ to BD (Fig. 302), it is obvious that $PK = LD$. Hence the result will follow if it can be shown that $PH = BL$.

Perhaps then $\triangle PBH \equiv \triangle BPL$.†

We have not as yet used the fact that $AB = AC$. This gives $\angle B = \angle C$. Obviously, by parallels, $\angle LPB = \angle C$. Thus $\angle B = \angle LPB$.

It is now easy to prove that $\triangle PBH \equiv \triangle BPL$, and the chain of reasoning is complete.

Cons.—Draw $BD \perp AC$, and $PL \perp BD$.

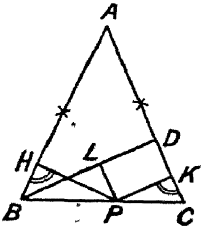

FIG. 302.

Proof.—∵ $\angle$s PLD, LDC are both right angles,
∴ (using LD as trv.) $PL \parallel DC$. P.2.
Hence (using BC as trv.) $\angle LPB = \angle C$. P.2*c*.
In $\triangle ABC$, ∵ $AB = AC$ (*Hyp.*),
∴ $\angle C = \angle HBP$. T.2.
Hence $\angle LPB = \angle HBP$.

In $\triangle$s HBP, LPB, ∵ $\left\{\begin{array}{l} BP = PB \\ \angle HBP = \angle LPB \\ \angle BHP = \angle PLB \end{array}\right\}$ ∴ $PH = BL$. C.2.

∵ $\angle$s LDK, PKD are both right angles,
∴ (using DK as trv.) $LD \parallel HK$. P.2.
But $LP \parallel DK$; ∴ $LPKD$ is $\parallel^{m}$; and $PK = LD$. Pm.1.

Hence $PH + PK = BL + LD = BD$,
that is $PH + PK$ is constant.

---

† Up to this point the analysis consists entirely of working *back* from the required conclusion.

NOTE.—In this rider the value of $PH+PK$ is determined by putting B into a *special* position, viz. at B. In the next rider this method will *not* be used.

Example 2.—*Two circles ABC, ABD intersect at two points A and B.* (Fig. 303) *P is a point on the arc ADB, such that the chords PA and PB when produced meet the arc ACB at points L and M. Prove that the chord LM is constant in length for all positions of P.*

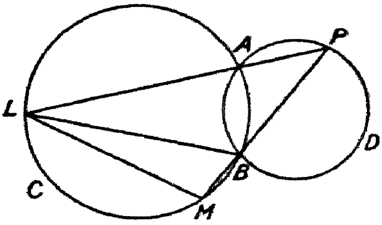

FIG. 303.

Analysis.—As P moves on arc ADB, L and M move on the arc ACB.

For all positions of P, $\angle APB$ is constant. AC.2.

Also for all positions of L and M, $\angle ALB$ and $\angle AMB$ are constant. AC.2.

Chord LM is constant, provided it subtends a constant angle at the centre, Ch.5.

i. e. provided it subtends a constant angle at a point on $\bigcirc^{ce}$, A.C.1.

i. e. provided $\angle LBM$ is constant.

But $\angle LBM = \angle BPL + \angle BLP$. T.1.

Also both $\angle BPL$ and $\angle BLP$ are constant.

Thus $\angle LBM$ is constant.

The chain of reasoning is now complete.

Cons.—Join LB.

Proof.— $\because$ $\angle APB$ is an angle in the fixed segment ADB,

$\therefore$ $\angle APB$ is constant. AC.2.

$\because$ $\angle ALB$ is an angle in the fixed segment ACB,

$\therefore$ $\angle ALB$ is constant. AC.2.

In $\triangle LPB$, ext. $\angle LBM = \angle ALB + \angle APB$. T.1.

Hence $\angle LBM$ is constant.

$\because$ chord LM subtends a constant angle at $\bigcirc^{ce}$ of fixed $\odot$ ABC,

$\therefore$ chord LM subtends a constant angle at the centre of fixed $\odot$ ABC; AC.1.

Hence chord LM is constant. Ch.5.

NOTE.—It is often of service to consider limiting positions of the lines in the data. Thus if P move along the circumference to A the line PA ultimately becomes the tangent at A to the $\odot$ ABD. If this tangent meets the circle ABC in T, then since PB becomes AB when P is at A, the line LM becomes TB. It would therefore have been sufficient to have proved LM equal to TB. In this case this is no easier than the original rider; but the method suggested will often be of service in other riders.

EXERCISES LXXI.

Riders.

1. Two circles cut at A and B and CAD, EAF are straight lines cutting the circles at C, D and E, F respectively. Show that $\angle CBE = \angle DBF$. AC. 2.

2. $ABCD$ is a quadrilateral inscribed in a circle. Show that if any other circle is drawn passing through A and B, and cutting AD, BC (or these lines produced) in P and Q, then PQ is parallel to CD. AC. 3.

3. A, B are two fixed points, and P, Q are two moving points. If the straight lines AP, BQ are always equal and parallel, prove that the distance PQ is constant. Pm. 2.

4. A, B are two fixed points, and P, Q are two moving points. If the straight lines AP, BQ always bisect each other, prove that the distance PQ is constant. C. 1.

5. Two circles intersect at P and Q, and a straight line AQB meets the circumference at points A and B on opposite sides of Q. Prove that $\angle APB$ is constant. AC. 2.

6. Two circles touch one another externally, and through the points of contact is drawn any straight line terminated by the circumferences; show that the tangents at the ends of this straight line are parallel. T. 1; Tn. 2.

7. The arcs AB, CD in the circle $ABCD$ are equal; show that the triangles ABC and DCB are equal in all respects. AC. 6*c*.

8. Two circles intersect at A and B, and at A tangents are drawn, one to each circle, to meet the circumferences in P and Q; show that the triangles ABP and ABQ are equiangular. AC. 5.

9. If two circles touch one another either internally or externally at the point P, and if two chords, PRr, PSs, be drawn through P, prove that the chords RS and rs are parallel. P. 2; AC. 5.

10. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn; show that the chords PX and QY are equal. Ch. 5; AC. 1.

11. If the two opposite sides of a quadrilateral inscribed in a circle are equal, the other two sides will be parallel. Ch. 5; AC. 1.

12. Two circles have external contact at A, and the line $BCDE$ is drawn to cut the circles; prove that the angles BAE, CAD are supplementary. AC. 5.

13. A secant OAB and a tangent OP are drawn to a circle from an external point O, and the bisector of the angle APB meets AB in C; prove that OC is equal to OP. AC. 5.

14. Two segments of circles are described on the same chord and on the same side of it; the extremities of the common chord are joined to any point on the arc of the exterior segment; show that the arc intercepted on the inner segment is constant. AC.2; AC.6.

---

15. A point *A* moves along a fixed line *PQ* and is joined by a straight line to a fixed point *R*. If *X* is the middle point of *AR*, show that the distance of *X* from *PQ* is constant. T.7.

16. *A* is a fixed point; *B*, *C* are two moving points on a fixed line *PQ*; *D*, *E* are the middle points of *AB*, *AC*. If *BC* is constant, prove that *DE* is constant. T.7.

17. In an acute-angled triangle *ABC* the lines drawn from *BC* perpendicular to the opposite sides intersect at a point *O*. If the angle *A* is constant, prove that the angle *O* is constant. T.1.

18. Two circles touch externally at *A*; if *BAC* and *DAE* are any two chords through *A* cutting the circles in *BC* and *DE* respectively, show that the triangles *ABD* and *ACE* are equiangular. AC.5.

19. Two circles intersect at points *A* and *B*, and they touch the line *PQ* at the points *P*, *Q*. Prove that the angles *PAQ*, *PBQ* are supplementary. AC.5.

20. Two equal circles intersect at *P* and *Q*, and from any point *A* on the circumference of one circle a straight line *APB* is drawn meeting the circumference of the other circle in *B*; prove that the lines *AQ* and *BQ* are equal. Ch.5.

21. The two tangents drawn to a given circle from a moving point form a constant angle. Show that the distance of this point from the centre is constant. Tn.2.

22. *ABC* is a triangle whose angles *B*, *C* are bisected by two lines meeting at *O*. If the angle *A* is constant prove that the angle *O* is constant. T.1.

23. The straight lines which bisect any angle of a quadrilateral figure inscribed in a circle and its opposite exterior angle meet on the circumference of the circle. AC.2; AC.3.

24. If two circles touch internally and a straight line is drawn to cut them, the segments of it intercepted between the circumferences subtend equal angles at the point of contact. AC.5.

25. If *AB*, *AC*, *AD*, be chords of a circle such that the angles *BAC*, *BAD* are supplementary, then the sum of the arcs on which these angles stand shall equal the whole circumference. AC.3; AC.6.

---

26. Find the radius of the circle described about any given right-angled triangle.

27. If a circle can be described about a parallelogram, prove that the parallelogram must be rectangular.

28. Given the base of a triangle (in magnitude and position) and the sum of the base angles, find the locus of the vertex.

29. Two equal circles intersect at *A* and *B*, and *BC, BD* are chords drawn in each circle so that each is equal to *AB*, show that *CA, DA* touch the circles.

30. Let *ABCD* be a quadrilateral inscribed in a circle, and let *AD* and *BC* meet in *E;* then shall the angle *AEB* be equal to the difference of the angles *ACB* and *DBC*.

31. *ABC* is an isosceles triangle, and on *BC* a segment of a circle is described cutting the equal sides in *D* and *E ;* prove that the line *DE* will be parallel to *BC*.

32. *ABC* is any triangle, and on the base *BC* a segment of a circle is described cutting the sides *AB, AC* in *D* and *E ;* prove that the triangles *ABC* and *ADE* are equiangular.

33. Given the base and vertical angle of a triangle, show that the radius of the circumscribing circle is constant.

34. There are two given segments of circles on the same side of the same base *AB*. If two points *P, Q* move one on each arc in such a way that *APQ* is always a straight line, prove that the angle *PBQ* is constant.

35*. *ABC* is a given triangle, and on the base *BC* a segment of a circle is described cutting the sides *AB, AC* in *D* and *E ;* prove that the line *DE* is always parallel to a fixed straight line.

36*. Two tangents *AP, AQ* are drawn to a given circle, and *R* is the middle point of the minor arc *PQ ;* prove that *PR* bisects the angle *APQ*.

37*. Two equal circles touch externally, and from the point of contact two chords are drawn at right angles to one another, one in each circle ; show that the line joining their other extremities is equal and parallel to the line joining the centres.

38*. *AB, AC* are two tangents to a given circle, and *D* is any point on the major arc cut off by *BC ;* prove that the sum of the angles *ABD* and *ACD* is constant.

39*. A quadrilateral figure is inscribed in a circle, and two of its opposite angles are bisected by straight lines which meet the circumference in *E* and *F;* prove that the line *EF* is a diameter of the circle.

40*. Four circles which have for chords the sides of a cyclic quadrilateral will intersect again in four concyclic points.

41*. *X* and *Y* are any two points taken on two arcs described on the same straight line *PQ*, and on the same side of it ; show that, if the bisectors of the angles *XPY* and *XQY* meet in *O*, then *PQ* subtends a constant angle at *O*.

42*. *ABC* is a triangle inscribed in a given circle ; if the angle *BAC* be constant, prove that the side *BC* always touches a concentric circle.

43*. The base of a given isosceles triangle is produced in either direction. If a point moves on this produced portion of the base, prove that the difference of its distances from the sides is constant.

44*. From any point within an equilateral triangle perpendiculars are drawn to the three sides. Show that the sum of these three perpendiculars is constant. [Use the result of Ex. 1, § 199.]

CHAPTER XIII.

THE CIRCLE: PROBLEMS.

200. PROBLEM.—**To bisect a given arc.**

Required to bisect the arc *ABC* (Fig. 304).

Cons.—Join the extremities of the arc by the line *AC*.

Bisect *AC* at right angles by the line *DE*, meeting the arc at the point *E*. § 113.

Then *E* shall bisect the arc *ABC*.

Proof.—Join *AE*, *CE*.

∵ *DE* bisects *AC* at right angles,

∴ *EA* = *EC*. L.3.

In ⊙ *ABC*, ∵ chord *EA* = chord *EC*,

therefore arc *EA* = arc *EC*. AC.6.

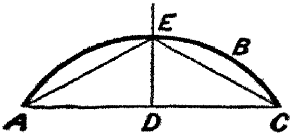

FIG. 304.

NOTE.—In practice we can find the line which bisects *AC* at right angles *without joining AC*. See Problem IV and Fig. 36 on p. 26.

201. PROBLEM.—**Given an arc of a circle, to find its centre.**

Required to find the centre of the circle of which *ABC* is an arc (Fig. 305).

Cons.—Take any three points *A*, *B*, *C* on the arc.

Join *AB*,† *BC*.† Bisect *AB*, *BC* at right angles by the lines *DE*, *FG*, meeting in *O*. § 113.

Then *O* shall be the centre of the circle.

Proof—∵ *DE* bisects the chord *AB* at right angles,

∴ *DE* passes through the centre. Ch.2.

Similarly *FG* **passes through the** centre. Ch.2.

Hence the centre is the point *O*, where *DE* and *FG* meet.

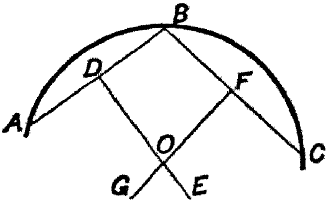

FIG. 305.

NOTE.—For accurate work *A* and *C* should be the extremities of the arc, and *B* near the middle point, if the arc is less than a semi-circle. If the arc is greater than a semi-circle, *A* and *C* should be roughly the extremities of a diameter, and *B* near the middle point of the arc *AC*.

---

† These lines are required in the proof only; for the lines *DE*, *FG* can be drawn without drawing the lines *AB*, *BC*.

EXERCISES LXXII (Practical).

1. The radius of an arc is 5 cms. ; the chord of the arc is 8 cms. Draw the arc, given that it is less than half the circumference, and measure the chord of half the arc.

2. Draw an arc whose radius is 2″, whose chord is 3″, and which is greater than half the $\bigcirc^{ce}$. Measure the chord of half the arc.

3. Draw a line AB 3″ long ; from A and B draw lines AC, BD, each 1″ long, making $\angle CAB = \angle DBA = 60^\circ$. Draw a circle passing through the four points A, B, C, D, measure the distance of the centre from AB.

4. Draw a figure in the shape of a capital **Z** ; make each angle 37°, the two parallel lines 2·5 and 3·9 cms., and the transversal 4 cms. Draw one circle to pass through both extremities of all three lines. Measure the distance of its centre from the shortest of the three lines.

202. *Prel. Ex.*—On a given circle find two points at each of which a given line subtends a right angle.

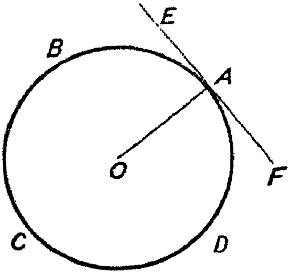

FIG. 306.

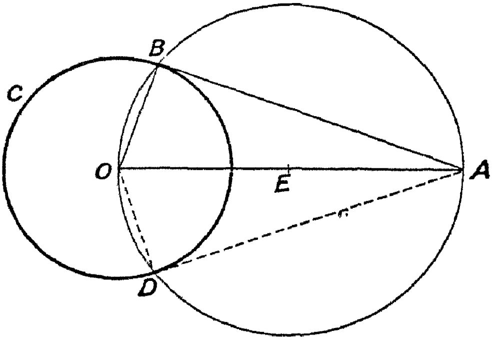

FIG. 307.

PROBLEM.—**From a given point to draw a tangent to a given circle.**

In Figs. 306 and 307 :—*Required* to draw a tangent to the given circle BCD, from the given point A.

CASE I.—When A is on $\bigcirc^{ce}$ of $\odot BCD$ (Fig. 306).

Cons.—Find O, the centre of $\odot BCD$.† § 201.
Join OA; through A draw $EAF \perp OA$. § 112.
Then EF shall be a tangent to the given $\odot$.

Proof.—In $\odot BCD$, $\because EAF \perp$ the radius OA,
therefore EAF touches the circle BCD. Th. 1.

---

† Usually the centre is known.

CASE II.—A lies outside $\bigcirc^{ce}$ of $\odot\ BCD$ (Fig. 307).

Cons.—Find O the centre of $\odot\ BCD$.† § 201.

Join OA, and bisect it at the point E. § 113.

With centre E and radius EA describe $\odot\ ABD$,‡ meeting the given $\odot$ at B and D. Join AB and AD.

Then AB and AD shall be tangents to the given circle.

Proof.—Join OB, OD.

$\because EO = EA =$ radius of $\odot\ ABD$, (*Cons.*) $\therefore O$ lies on $\bigcirc^{ce}$ of $\odot\ ABD$.

$\because OA$ is a diameter of $\odot\ OAB$, $\therefore \angle OBA = \text{R}$. AC.4.

In $\odot\ BCD$, $\because AB \perp$ radius OB,
therefore AB touches $\odot\ BCD$.

Similarly it is easily proved that $AD \perp OD$,
and hence that AD touches $\odot\ BCD$. Tn.1.

203. DEFINITION.—**If a line touches each of two circles it is called a** *common tangent* **of those circles.**

If the two circles lie on the same side of the line, the line is called a **direct** common tangent; if the circles lie on opposite sides of the line it is called a **transverse** common tangent.

Thus in Fig. 308 PQ is a direct common tangent of the two circles, and in Fig. 309 PQ is a transverse common tangent of the two circles.

Prel. Ex.—(i) How many direct and how many transverse common tangents can be drawn to two circles (a) if each circle lies wholly outside the other, (b) if the two circles touch externally, (c) if the two circles cut one another, (d) if the two circles touch internally, (e) if one circle lies wholly within the other?

(ii) A and B are two given points. Show how to find a point C which shall be at a given distance from A, and at which the line AB subtends a right angle.

(iii) PQ is a direct common tangent of two circles, touching the larger circle (centre A) at P, and the smaller circle (centre B) at Q; BC is drawn perpendicular to AP. Prove that AC is equal to the difference of the radii of the two circles.

(iv) Show how to find the point C in Question iii given only the two circles (compare Question ii). Hence show how to construct the direct common tangent.

(v) If, in Question iii, PQ is a transverse common tangent, prove that AC is equal to the sum of the radii of the two circles. Hence show how to construct the transverse common tangent

---

† Usually the centre is known.

‡ Only small arcs of this circle including the points B and D are required for the *construction*.

PROBLEM.—**To draw the direct common tangents to two given unequal circles.**

In Fig. 308:—*Required* to draw the direct common tangents to the two given unequal circles *PXR, QYS.*

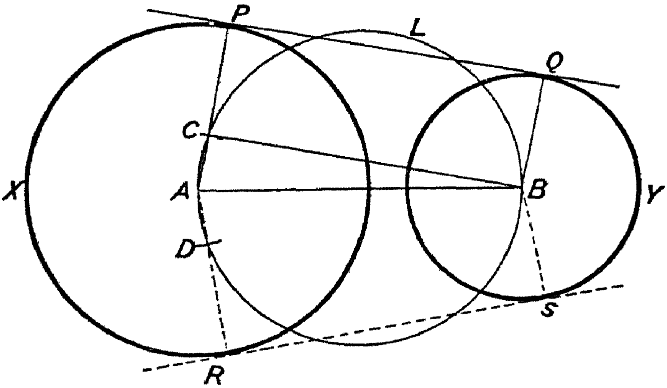

FIG. 308.

Cons.—Find *A*, *B*, the centres of the two circles.† § 201. Let *A* be the centre of the larger circle. On diameter *AB* describe ⊙ *ABL*.

With centre *A* and radius equal to the difference of the radii of the two given circles describe two arcs cutting ⊙ *ABL* at the points *C, D.*

Join *AC* and produce it to meet the ○ᶜᵉ *PXR* at *P*. Draw *BQ*, ∥ to, and in the same sense as *CP* meeting the ○ᶜᵉ *QYS* at *Q*. Join *PQ*, *BC*. § 111.

Then *PQ* shall be a direct common tangent.

Proof.—∵ *AC* is equal to the difference between the radii of the two circles (*Cons.*), and *AP* is the radius of the larger circle,
∴ *CP*=radius of smaller ⊙ ; *i. e.* *CP*=*BQ*.
∵ *BQ*=*CP*, and *BQ* ∥ *CP*, ∴ *BQPC* is ∥ᵐ. Pm.2.

∵ *ALB* is a semi-circle, ∴ ∠*ACB*=R (AC. 4); whence ∠*PCB*=R.
∵ ∠*PCB*=R, ∴ ∥ᵐ *BQPC* is a rect. (Pm.3); *i.e.* ∠*BQP*= ∠*CPQ*=R.

In ⊙ *QYS*, ∵ *PQ* ⊥ radius *BQ*, ∴ *PQ* touches ⊙ *QYS*. Tn.1.
In ⊙ *PXR*, ∵ *PQ* ⊥ radius *AP*, ∴ *PQ* touches ⊙ *PXR.* Tn.1.

---

† Usually the centres are known.

Also both circles lie on the same side of *PQ*.

Hence *PQ* is a direct common tangent to the two circles.

In the same way by making the same construction on the lower side of *AB* a second direct common tangent, *RS*, can be drawn.

204. PROBLEM.—**To draw the transverse common tangents to two given circles.**

In Fig. 309:—*Required* to draw the transverse common tangents to the two circles *PXR, QYS*.

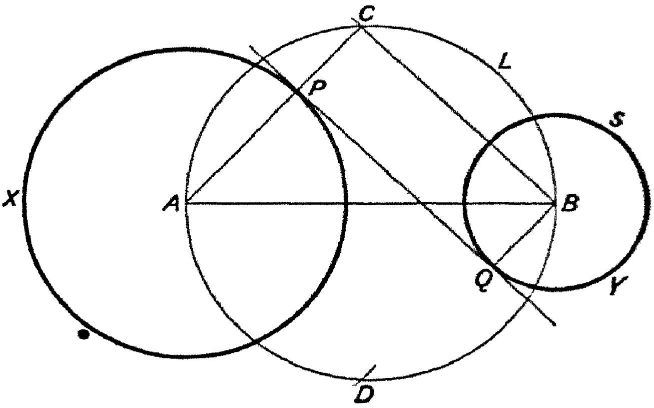

FIG. 309.

Cons.—Find *A*, *B*, the centres of the two circles. § 201.

On diameter *AB* describe ⊙ *ABL*.

With centre *A* and radius equal to the sum of the radii of the two given circles describe arcs cutting ⊙ *ABL* at the points *C*, *D*. Join *AC*, meeting ○^{ce} *PXR* at *P*.

Draw *BQ*, ∥ to, and in same sense as *CP*, meeting ○^{ce} *QYS* at *Q*. Join *PQ*, *BC*. § 111.

Then *PQ* shall be a transverse common tangent.

Proof.—Following the same argument as in the preceding Problem, it is easy to prove that *CP*=*BQ*, and that *BQPC* is a rect.

Hence *PQ* is a tangent to both circles.

In the same way by using the point *D*, instead of the point *C*, a second transverse common tangent can be drawn to the two circles.

205. PROBLEM.—**To circumscribe a circle to a given triangle.**

Required to circumscribe a circle to the triangle ABC (Figs. 310, 311, 312).

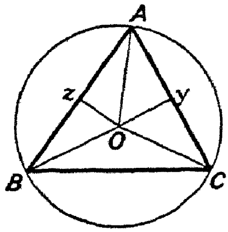

FIG. 310.

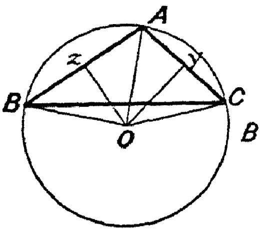

FIG. 311.

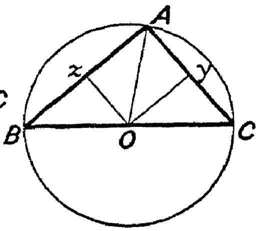

FIG. 312.

Cons.—Bisect the sides AB, AC at right angles by lines zO, yO meeting at O. § 113.

With centre O and radius OA, describe a circle. This circle shall pass through B and C.

Proof.—Join OA, OB, OC.
$\because$ the line zO bisects AB at right angles, $\therefore$ $OA = OB$. L.3.
$\because$ yO bisects AC at right angles, $\therefore$ $OA = OC$. L.3.
Hence the circle described with centre O and radius OA passes through both B and C;
hence this circle is circumscribed to $\triangle ABC$.

NOTE.—The point O is called the **circum-centre** of $\triangle ABC$; it may lie within the triangle (Fig. 310), or in one side (Fig. 311), or without the triangle (Fig. 312). The radius OA is called the **circum-radius** of $\triangle ABC$, and the circle is called the **circum-circle.**

206. *Prel. Ex.*—AB is a straight line, AC and BD are two other straight lines on the same side of AB:—

(i) Find the locus of the centre of a circle touching both arms of $\angle B$. (ii) Find the locus of the centre of a circle touching both arms of $\angle C$. (iii) Find the centre of a circle touching each of the three lines AB, AC, BD.

PROBLEM.—**To draw a circle touching three given lines which are neither concurrent nor all parallel.**

In Fig. 313:—

Required to draw a circle touching each of the three lines AB, CD, EF.

Cons.—Since the three lines are neither concurrent nor all parallel, at least one of the three lines will intersect the other two lines at different points.

Let EF meet the other two lines at the points G, H, and let BGH, DHG be the two interior angles formed on one side of the line EF.

Bisect these two angles by lines GK, HL, which will meet at some point O; § 107.

draw OM, ON, OP, $\perp$ to GH, GB, HD respectively. § 114.

With centre O and radius OM draw a circle; this shall be the circle required.

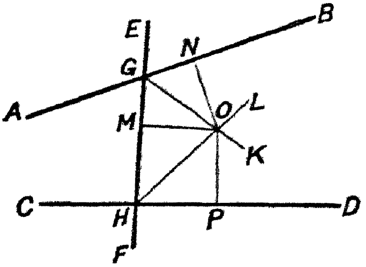

FIG. 313.

Proof.—(i) To prove that the lines GK, HL will meet:—

$\because$ each of angles BGH, DHG is less than 2R;

$\therefore \angle BGH + \angle DHG < 4R$,

$\therefore \frac{1}{2}\angle BGH + \frac{1}{2}\angle DHG < 2R$,

i.e. $\angle KGH + LHG < 2R$.

Hence GK and HL are not parallel. P.2*c*.

(ii) To prove that the circle with centre O and radius OM will touch all three lines:—

$\because OG$ bisects $\angle BGH$, $\therefore$ perpendicular $ON =$ perpendicular OM. L.4.

$\because OH$ bisects $\angle DHG$, $\therefore$ perpendicular $OP =$ perpendicular OM. L.4.

$\because OM$, ON, OP are all equal,

$\therefore$ the circle with centre O and radius OM passes through N and P.

In $\odot$ MNP, $\because AB$, CD, EF are respectively perpendicular to the radii ON, OP, OM,

therefore AB, CD, EF are tangents to this circle. Tn.1.

207. Incircles and Excircles.—

DEFINITION.—**A circle which touches the three sides of a triangle and lies within the triangle is said to be inscribed in the triangle.**

The inscribed circle of a triangle is often called its **incircle**. Its centre is called the **incentre** and its radius the **inradius**.

DEFINITION.—**If a circle touches one side of a triangle and the other two sides produced it is called an** *escribed circle* **(or** *ex-circle*) **of the triangle. The centre of the circle is called an** *ex-centre* **of the triangle, and its radius an** *ex-radius*.

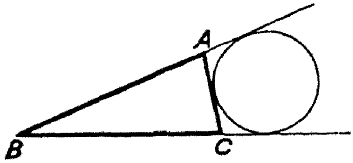

FIG. 314.

Fig. 314 represents a circle escribed to a triangle *ABC*.

EXERCISES LXXIII.

PRACTICAL GEOMETRY.

1. Draw a circle of radius 2·5″. Draw two radii inclined at an angle of 56°. Draw the tangents at the extremities of these radii; measure the distance of the point of intersection of these tangents from the centre of the circle.

2. Draw a circle of radius 1·8″. Take a point at a distance of 3″ from the centre. Draw the two tangents from this point to the circle, and measure the distance between their points of contact.

3. Draw two circles of radii 1 and 3 cms., their centres being 5 cms. apart. Draw a direct common tangent to these circles and measure the distance between the two points of contact.

4. Draw two circles of radii 1·5 and 3 cms., their centres being 5 cms. apart. Draw a transverse common tangent, and measure the distance between the two points of contact.

5. If any three lines are drawn in a plane, then either (i) they are all parallel, (ii) two of them are parallel, (iii) they are concurrent, (iv) they form a triangle. In each case how many circles can be drawn to touch all three lines ?

6. Two parallel lines are drawn at a distance of 2·4 cms. from each other; a transversal is drawn crossing them at any angle. Find, without accurate drawing, the lengths of the radii of all the circles which can be drawn to touch all three lines.

7. Draw a triangle whose sides are 3, 2·5, 2 cms. respectively. Draw the incircle, the circumcircle, and the three excircles to this triangle, and measure their radii.

8. Draw an equilateral triangle of side 1″. Find the lengths of the inradius, circumradius, and exradius.

9. Draw a triangle whose sides are 3, 4, 5 cms. Find the lengths of the inradius, circumradius, and exradii.

10. The circumradius of a triangle is 1·5″, its base is 2·4″, and one of its base angles is 48°. Find the sum of its three sides.

RIDERS.

11. Show that it is not possible to draw a circle through three points lying in the same straight line. L.3; P.2.

12. If the circumcentre of a triangle lies on one side of the triangle, show that the angle opposite to this side is a right angle. T.1; T.2.

13. If the circumcentre of a triangle lies outside the triangle, show that one angle of the triangle is obtuse. T.1; T.2.

14. If the circumcentre of a triangle lies within the triangle, show that all the angles of the triangle are acute. T.1; T.2.

208. PROBLEM.—**From a given circle to cut off a segment to contain a given angle.**

In Fig. 315:—

Required to cut off from ⊙ ABC a segment to contain an angle equal to X.

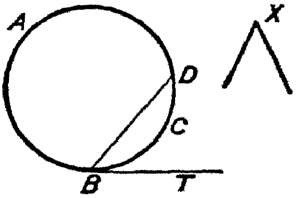

FIG. 315.

Cons.—At any point B on $\bigcirc^{ce}$ draw tangent BT. § 202.

At B draw the chord BD, making $\angle TBD = \angle X$. § 109.

Then shall the segment BAD, which is alternate to the angle TBD, be the segment required.

Proof.—$\because$ BT is a tangent to the circle, and BD a chord,

$\therefore$ $\angle TBD = \angle$ in segment BAD. AC.5.

But $\angle TBD = \angle X$.

Hence angle in segment $BAD = \angle X$.

NOTE.—The following is the simplest method, *if it works*:—Draw a chord AB; draw a second chord AC making $\angle BAC$ equal to given angle. Segment BAC is required segment.

Why does not this method always work? What rule can you frame to ensure that it should work?

209. *Prel. Ex.*—Draw an acute angle BAC. It is required to draw a segment of a circle on base AB, on the opposite side to the point C, to contain an angle equal to $\angle BAC$:—

(i) AB is a chord of the required circle; what do you know about AC?
(ii) Find two lines which intersect at the centre of the required circle.
(iii) Repeat Exercises i, ii, taking BAC as an obtuse angle.

PROBLEM.—**On a given base to draw a segment of a circle to contain a given angle.**

In Figs. 316, 317, 318:—

Required to describe, on base AB, a segment of a circle to contain an angle equal to the angle C.

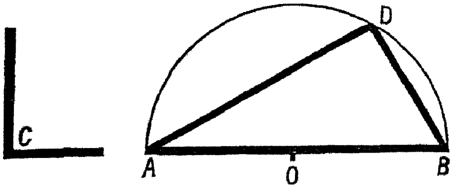

FIG. 316.

CASE I.—If $\angle C = R$ (Fig. 316):—

Cons.—On AB describe a semi-circle ADB.

Proof.—Then angle in segment $ADB = R$. AC. 4.

CASE II.—$\angle C \neq R$ (Figs. 317, 318):—

Cons.—At A make $\angle BAF = \angle C$. § 109.
At A draw $AE \perp AF$. § 112.

Bisect AB at right angles by the line DO meeting AE at O. § 113.

With centre O and radius OA describe the circle AGE. This circle shall pass through B, and segment AGB shall be the segment required.

Proof.—Join OB.
$\because$ DO bisects AB at right angles, $\therefore$ $OB = OA$. L. 3.
Hence $\odot$ AGE passes through the point B.
$\because$ $AF \perp$ radius OA at its extremity A,
$\therefore$ AF is a tangent to $\odot$ AGB. TH. 1.

In $\odot$ AGB, $\because$ AF is a tangent and AB a chord,
$\therefore$ $\angle BAF =$ angle in alternate segment AGB. AC. 5.
But $\angle BAF = \angle C$.
Hence angle in segment $AGB = \angle C$.

DEFINITION.—**The *height* (or *altitude*) of a segment of a circle is the length of the line drawn perpendicular to the base of the segment from its middle point and terminated by the arc of the segment.**

Thus in Fig. 304 *DE* is the height or altitude of the segment *ABC*.

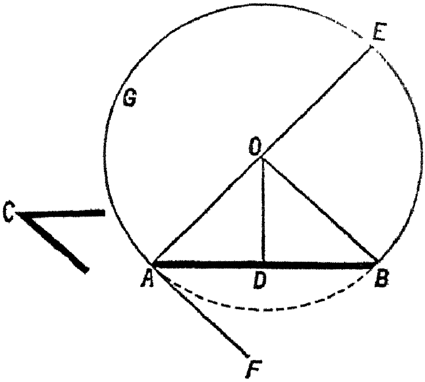

FIG. 317.

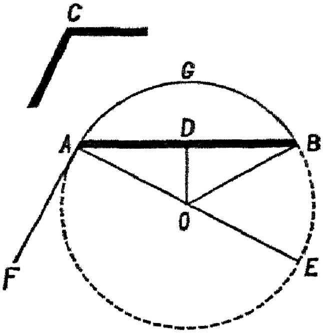

FIG. 318.

EXERCISES LXXIV (Practical).

1. Draw a circle of radius 1·4″. From this circle cut off two segments to contain angles of 54° and 126° respectively. Measure the altitudes of these segments.

2. On a base of 1″ describe a segment of a circle to contain an angle of 35°. Measure the height of the segment.

3. Draw the locus of the vertex of a triangle on a base of 6 cms., having a vertical angle of 90°. Draw the locus of the vertex of a triangle on the same base whose altitude is 2 cms. Hence draw a triangle on base 6 cms., whose vertical angle is 90°, and whose altitude is 2 cms. Measure the sides of this triangle.

4. Draw the locus of the vertex of a triangle on a base 6 cms. whose vertical angle is 120°. Draw the locus of the vertex of a triangle on the same base whose median is 2 cms. Hence draw a triangle whose base is 6 cms., whose median is 2 cms., and whose vertical angle is 120°. Measure its sides.

5. Draw a triangle *ABC*, in which $AB = 1''$, $AC = 1{\cdot}6''$, $\angle A = 97°$. Find a point *O*, within the angle *BAC*, at which *AB* subtends an angle of 60°, and *AC* subtends an angle of 106°. Measure *OA*. [Use two intersecting loci.]

210. *Prel. Ex.*—(i) ABC is a triangle inscribed in a circle, and DAE is the tangent to this circle at A. Prove that the three angles at A are respectively equal to the three angles of the triangle.

PROBLEM.—**In a given circle to inscribe a triangle equiangular to a given triangle.**

In Fig. 319:—*Required* to inscribe in $\odot\ ABC$ a triangle equiangular to $\triangle\ DEF$.

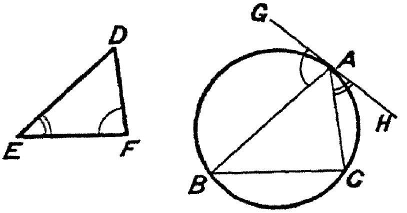

FIG. 319.

Cons.—At any point A on $\odot\ ABC$ draw tangent GAH. § 202.

From A draw the chord AB, making $\angle GAB = \angle F$. § 109.

From A draw the chord AC, making $\angle HAC = \angle E$. § 109.

Join BC. Then ABC shall be the required triangle.

Proof.—$\because$ AC is a chord, and AH a tangent of $\odot\ ABC$,
$\therefore\ \angle HAC = \angle B$. AC.5.
But $\angle HAC = \angle E$;
hence $\angle B = \angle E$.

Similarly $\angle GAB = \angle C$, and $\angle GAB = \angle F$;
hence $\angle C = \angle F$.

Add these equals, thus $\angle B + \angle C = \angle E + \angle F$.
But in $\triangle$s ABC, DEF, $\angle BAC + \angle B + \angle C = \angle D + \angle E + \angle F$; T.1.
hence, subtracting equals from equals, $\angle BAC = \angle D$.

211. *Prel. Ex.*—(i) A triangle is circumscribed to a circle, and the three radii are drawn which pass through the points where the sides of the triangle touch the circle. Show that the three angles between these radii are respectively supplementary to the three angles of the triangle.

PROBLEM.—**To circumscribe to a given circle a triangle equiangular to a given triangle.**

In Fig. 320:—*Required* to circumscribe to $\odot\ ABC$ a triangle equiangular to $\triangle\ DEF$.

Cons.—Produce EF, in both directions, to G and H.
Find O the centre of ⊙ ABC. § 201.
Draw any radius OA.
Draw two other radii OB, OC on opposite sides of OA, making $\angle AOB = \angle DEG$, and $\angle AOC = \angle DFH$. § 109.

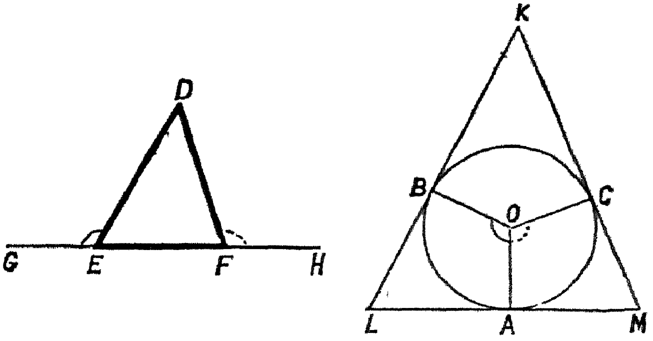

FIG. 320.

Draw the tangents to the circle at A, B, and C and produce them to form △ KLM. § 202.

Then KLM shall be the required triangle.

Proof.—∵ OA, OB, OC are the radii drawn to the points of contact of the three tangents LM, LK, KM,

∴ these radii are respectively ⊥ these tangents. Th. 1.

In quad. $AOBL$, the four angles are together equal to 4 R. T. 1.

But the two angles at A and B are right angles;

hence $\angle AOB + \angle L = 2\text{R}$.

∵ GEF is a straight line, ∴ $\angle DEG + \angle DEF = 2\text{R}$; A. 1.

∴ $\angle AOB + \angle L = \angle DEG + \angle DEF$.

But $\angle AOB = \angle DEG$; *Cons.*

hence, subtracting equals from equals $\angle L = \angle DEF$.

Similarly it can be shown that, in quad. $AOCM$, $\angle AOC + \angle M = 2\text{R}$, and hence that $\angle AOC + \angle M = \angle DFH + \angle DFE$;

whence $\angle M = \angle DFE$.

Add these equals; ∴ $\angle L + \angle M = \angle DEF + \angle DFE$.

But in △s KLM, DEF, $\angle K + \angle L + \angle M = \angle D + \angle DEF + \angle DFE$; T. 1.

hence, subtracting equals from equals, $\angle K = \angle D$.

G. T. P.

212. **On the Solution of Problems.**—The methods for solving riders, and the special methods which should be applied when the riders are problems, have already been explained at length in §§ 144, 150. The following examples are good illustrations of these methods as applied to riders on the circle.

FIRST METHOD.—*Intersection of Loci.*

Example.—*ABC is a given angle. Show how to draw a circle which shall touch the arm AB at the given point A, and shall also touch the arm BC.* (Fig. 321.)

Analysis.—(i) It is obvious that the solution of the problem depends upon finding the position of the centre of the circle.

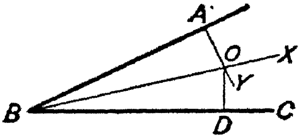

FIG. 321.

(ii) The circle is to touch both arms of the angle B. Hence its centre must be equidistant from the arms of this angle.
Therefore the centre must lie on *the line which bisects* $\angle B$. L. 4.

Again, the circle touches AB at A.
Hence its centre must lie on *the line through* A, which is $\perp AB$. Th. 1.

Hence the centre is the point of intersection of these two lines; and the required construction and proof are as follows:—

Cons.—Bisect $\angle B$ by the line BX.
At A draw $AY \perp$ to AB, meeting BX at O.
With centre O and radius OA describe a circle.
This shall be the circle required.

Proof.—Draw $OD \perp BC$.

$\because$ O lies on the line which bisects $\angle B$, and OA, OD are drawn perpendicular to the arms of this angle,

$\therefore$ $OA = OD$. L. 4.

Hence the circle passes through D.

$\because$ BA and BD are perpendicular to the radii OA, OD at their extremities A, D,
therefore BA and BD are tangents to the circle. Th. 1

SECOND METHOD.—*Inference from the required result.*

In this method the student should first draw the original figure (representing the problem solved) in ink, and any lines which he may add (for trial) in pencil, so that they are easily removed if wrong.

Considerable practice is needed before he will succeed in drawing the right lines; but it may be noted that if the middle point of a chord of a circle occurs in a problem it is almost certainly right to draw the line joining this point to the centre, so that the fresh fact indicated by Ch.1 may be brought out; if a tangent is named, the diameter through the point of contact should be drawn because of Tn. 1; if two circles, their line of intersection and the join of their centres; and so on.

Example.—*Through a point of intersection of two given circles draw a line terminated by the circumferences which shall be bisected at this point of intersection.*

Analysis.—Draw Fig. 322 to represent the problem as solved; so that CD is bisected at A.

The figure suggests no facts without further construction lines. Various constructions must be tried, till some fact of value is obtained.

If PH, QK be the perpendiculars from the centres, then $AH = \frac{1}{2} AC$, $AK = \frac{1}{2} AD$. Ch.1.

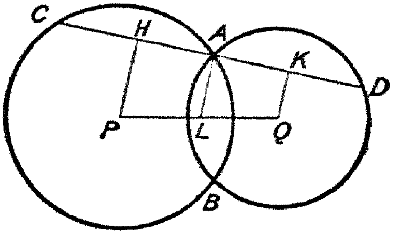

FIG. 322.

But $AC = AD$; thus A bisects HK.

Draw $AL \perp CD$; then HP, AL, KQ are parallel. Hence *L bisects PQ*. P.5.

This fact at once suggests the following construction:—

Cons.—Find P, Q the centres of the circles. Bisect PQ at L; join LA.
Draw $CAD \perp LA$. Then CAD shall be the required line.
Draw PH, $QK \perp CAD$.

Proof.—$\because$ PH, LA, QK are all $\perp CD$, $\therefore$ they are all parallel. P.2.
But $PL = LQ$; hence $HA = AK$. P.5.

$\because$ PH, QK are $\perp$ chords CA, AD, $\therefore$ $CA = 2\,HA$, and $AD = 2\,AK$. Ch.1.
Hence $CA = AD$.

NOTE.—In many easy problems it will be found that either method can be used, and that the two methods are practically the same. For instance, the example given under the first method can be solved as easily by the second method.

EXERCISES LXXV.

RIDERS.

1. Describe a circle of given radius to touch a given circle externally at a given point.

2. Describe a circle of given radius to touch a given circle internally at a given point.

3. Describe a circle of given radius to pass through two given points. When will there be (*a*) two solutions, (*b*) one solution, (*c*) no solution?

4. Describe a circle to pass through two given points and to have its centre on a given straight line.

5. Describe a circle with a given centre to touch a given circle (*a*) externally, (*b*) internally.

6. Describe a circle with a given centre to bisect the circumference of a given circle.

7. Describe a circle with given radius to touch two given circles externally.

8. Describe a circle through a given point, touching a given straight line at a given point.

9. Describe a circle through a given point, touching a given circle at a given point.

10. Describe a circle of given radius to touch a given circle and a given straight line. How many solutions are possible?

11. Through E, a given point within the circle ABC, show how to draw a chord which shall be bisected at E. Can more than one such chord be drawn?

12. Show how to describe a circle with a given point as centre, so as to cut off a given length from a given straight line.

13. Draw a tangent to a given circle which shall be (*a*) parallel, (*b*) perpendicular to a given straight line.

14. Draw a tangent to a given circle which shall make a given angle with a given straight line.

15. Show how to draw the direct common tangents to two equal circles.

16. Find a point at which each of two given finite straight lines subtends a given angle.

17. Show how to inscribe in a given circle a triangle equiangular to a given triangle, without drawing a tangent.

18. AB is a fixed chord in a circle; show how to inscribe in the circle an isosceles triangle having (*a*) AB for its base, or (*b*) AB for one of its two equal sides.

19. Through a given point within a circle draw the smallest possible chord.

20. Given the hypotenuse and the length of the perpendicular from the right angle upon it, construct the right-angled triangle.

21. Through a given point draw a chord of a given circle that shall be equal to a given straight line. Under what circumstances is this impossible?

22. In a given circle show how to draw a chord which shall be double of its distance from the centre.

23. Through a given point within a circle show how to draw two equal chords at right angles to one another.

24. Show how to draw through a given point a straight line which will cut off from a given circle a segment which shall contain a given angle.

25. Find the locus of a point the tangent from which to a given circle will be of given length.

26. Show how to find a point such that the two tangents from this point to a given circle will contain a given angle. Find the locus of all such points.

27. Show how to draw a circle to touch a given circle and to touch two intersecting tangents of the given circle.

28. An equilateral triangle is inscribed in a circle. If a tangent is drawn to the circle at each vertex of the triangle, show that these three tangents form an equilateral triangle. AC.5.

29. In an isosceles triangle prove that the inscribed circle and one of the escribed circles touch the base at its middle point. Tn.2.

30. **Show that the centres of two circles, the intersection of their direct common tangents, and the intersection of their transverse common tangents lie in one straight line.** Tn.2.

31. Given the base and the vertical angle of a triangle; find the locus of the point of intersection of the bisectors of the base angles. T.1; L.5.

32. [Tn.4*c*.]—**If the sum of one pair of opposite sides of a convex quadrilateral is equal to the sum of the other pair, then a circle can be inscribed in the quadrilateral.**

[If *ABCD* is the quadrilateral, suppose that the circle which touches *AB, BC, CD* does not touch *AD*. Draw the tangent from *A*, meeting *CD* at *E*; apply Tn.4 to quadrilateral *ABCE* and finish by "reductio ad absurdum."]

33. ABC, DCE are two perpendicular straight lines; also $AB=BC=CD=CE$. Two rods, which originally coincide in magnitude and position with the lines AC and DE, are pivoted the one at B and the other at C; each rotates counter-clockwise about its pivot at the rate of 1° per second. If they start at the same moment, find the locus of the point where the two rods cross.

CONSTRUCTION OF TRIANGLES AND QUADRILATERALS (*Riders*).

34. Construct an equilateral triangle given its altitude.

35. Construct a right-angled triangle having one acute angle double the other.

36. Construct a triangle given the base, the vertical angle, and the altitude.

37. Construct a triangle given the base, the vertical angle, and the median which bisects the base.

38. Construct an isosceles triangle given the base and the in-radius.

39. Construct a triangle given the base, one base angle, and the sum of the two sides.

40. Construct a triangle given the base, one base angle, and the difference of the two sides.

41. In a given square inscribe a square having one angular point at a given point in one side of the given square.

42. About a given circle describe a rhombus having one angle equal to a given angle.

43. Construct a triangle given the in-radius, one angle, and the distance of the vertex of that angle from the opposite side.

CALCULATIONS.

44. A man pulling a truck takes steps of 22 inches. If each wheel of the truck is 3 feet high, through what angle does each wheel turn at every step?

45. A workman is pulling up a basket of bricks to a height of 50 feet from the ground by means of a rope passing over a wheel fixed to the top of a scaffolding. If the diameter of the wheel is 10 inches how many turns does it make?

46. The length of an oar from the rowlock to the handle is 3 feet, and at each stroke the rower pulls it through an angle of 80°. How many miles does he move his hands to and fro in rowing for 20 minutes at the rate of 35 strokes a minute?

47. A see-saw is made with a plank balanced over the trunk of a tree. The tree-trunk is circular and is 2 feet thick. When the plank is horizontal its middle point is in contact with the trunk; when one end is pushed down to the ground the plank slopes at 30° to the horizontal. Find the length of the plank. (Solve partly by drawing and partly by calculation.)

Practical Geometry.

48. Draw a triangle whose sides are 2″, 3″ and 4″; draw the inscribed and circumscribed circles and measure their radii.

49. Using only the compasses and ruler draw three lines AB, BC, CD, making $BC = 2''$, $\angle B = 120^\circ$, $\angle C = 60^\circ$, A and D lying on the same side of BC. Draw a circle touching each of these three lines, and measure the parts into which BC is divided by the point where it touches this circle.

50. Construct a quadrilateral $ABCD$ making $AB = 1''$, $\angle A = \angle B = 130^\circ$, and such that AB and CD are parallel and 1·5″ apart. Draw a circle touching the three sides AD, AB, BC, and measure the length of the chord which this circle intercepts on the side CD.

51. The radius of a circle is 3 cms. and the length of the tangent drawn to this circle from a certain point is 2·5 cms. Find the distance of this point from the centre of the circle, and also from the circumference.

52. In a circle of radius 2 cms. inscribe an equilateral triangle. Measure its side.

53. In a circle of radius ·8″ inscribe a triangle having angles 50°, 60°, 70°. Measure its sides.

54. In a circle of radius 4 cms. inscribe a triangle having angles 30°, 40°, 110°. Measure its sides.

55. Describe an equilateral triangle about a circle of radius ·5″. Measure its side.

56. About a circle of radius 2 cms., describe a triangle whose angles are 48°, 60°, 72°. Measure its sides.

57. About a circle of radius ·5″ describe a triangle whose angles are 40°, 40°, 100°. Measure its sides.

58. In a circle of radius 1·5″ inscribe a triangle whose angles are 90°, 45°, 45°. Find the radius of the circle inscribed in this triangle.

59. Draw a circle of radius 1″. Draw two tangents to this circle including an angle of 70° and measure their lengths.

60. Draw a circle of radius 1·4″. Find a point such that the two tangents drawn to the circle from this point form an angle of 110°. Measure the distance of this point from the centre.

61. Draw a circle of radius 3 cms. Find a point from which each tangent to the circle measures 5 cms. Measure the angle between these tangents.

62. Draw an angle of 40°. Describe a circle of radius 1·5 cms. touching both arms of this angle. Find the distance of the centre from the angular point.

63. Draw a sector of a circle of radius $2''$, in which the radii include an angle of 60°. Inscribe a circle in the sector and measure its radius.

64. In a quadrant of a circle of radius 4 cms. inscribe a circle. Measure its radius. [A quadrant is a sector whose angle is a right angle.]

65. Draw a semicircle on a diameter of length 6 cms. Inscribe a circle of radius 1 cm. Measure the distance between the centre of the semicircle and the point where the circle touches the diameter of the semicircle.

66. Draw a circle of radius $\cdot 5''$, and a line at a distance of $1''$ from its centre. Draw two circles each of radius $\cdot 4''$, touching the given line and the given circle. Find the distance between the centres of these circles.

67. Draw a triangle ABC, making $AB = 1''$, $BC = 1\cdot 5''$, $CA = 2''$. Draw a circle through B touching AC at A. Measure its radius.

68. Draw a circle of radius 3 cms. Draw two radii OA and OB, making $\angle\ AOB = 30°$. Produce OA to C, making $OC = 5\cdot 2$ cms. Draw a circle through C touching the given circle at B, and measure its radius.

69. Draw three circles of radii 1, 1·5, 2 cms., each of which touches the other two externally. At the point where the largest circle touches the smallest draw the common tangent; measure the chord intercepted on this tangent by the third circle.

70. Draw three circles of radii 1, 2, 4 cms., of which the two small circles touch each other externally, and each touches the larger internally. Measure the distance from the centre of the smallest circle to the point of contact of the other two circles.

71. Draw two circles, each of radius 4 cms., their centres being 5 cms. apart. Draw 8 circles each of radius 1 cm., each touching the two given circles. Measure the distance of the centre of each of these small circles from the join of the centres of the two given circles.

72. Draw a line ABC, making $AB = 1''$, $BC = 1\cdot 7''$; draw a second line AD, making $\angle DAB = 32°$. Find two points on AD at which the line BC subtends an angle of 58°; measure the distance between these two points.

73. Construct two circles with radii 1·5 cms. and 2·7 cms., the distance between their centres being 3·4 cms. Draw their common tangents and measure their length.

74. Draw two circles of radii 2 and 1 cms. respectively, with centres 2 cms. apart. Draw the common tangents and measure the distance of their intersection from the centre of the smaller circle.

75. Draw two circles of radii 3 and 1 cm. respectively with centres 6 cms. apart. Draw the transverse common tangents and measure the distance of their intersection from the centre of the smaller circle.

76. Draw two circles of radii 2 and 3 cms. with their centres 2 cms. apart. Draw the common tangents and find the distance of their point of intersection from the circumference of the nearer circle.

77. Draw two circles of radii ·5″ and 1·5″, having their centres 2 6″ apart. Draw the common tangents, and find the distance between the point of intersection of the direct common tangents and that of the transverse common tangents.

78. Draw two circles of radii 2 and 3 cms. touching externally. Draw the direct common tangents, and measure the distance between the two points in which these tangents touch the larger circle.

79. Draw a circle of radius 1·5″. Take a point A at a distance of 2″ from the centre. Draw two lines through A, on each of which the circle intercepts a chord of length 1·8″. Measure the angle at A.

80. Draw a circle of radius 1″, and draw two tangents to this circle AB, AC making $\angle A = 40°$. Find the length of the radius of a circle which touches AB, AC, and the given circle. Also find the length of the radius of a circle which touches AB produced, AC produced, and the given circle.

81. Draw two circles of radii 1·2 and 2·7 cms. touching externally. Draw the triangle for which one of these circles is the inscribed circle and the other an escribed circle. Measure its sides.

82. Draw two circles of radii 1·2″ and ·5″, the least distance between the circumferences being ·4″, and each circle lying entirely without the other. Draw the triangle for which one of these is the inscribed and the other an escribed circle. Measure the sides of the triangle.

83. Draw two circles of radii 1·5″ and ·8″, having their centres 3″ apart. Draw the triangle for which these circles are both escribed circles. Measure the sides of the triangle.

84. Draw an equilateral triangle of side 1″. Circumscribe a circle to this triangle. Circumscribe a triangle to this circle, whose sides touch the circle at the vertices of the first triangle. Circumscribe a circle to this triangle. Measure the radius of the larger circle.

85. A man walking along a straight road observes that a church spire P is due North of him when he is at a certain point A. After walking three quarters of a mile to B he observes that the direction of the spire is at right angles to the road; and after walking half a mile farther to C he observes that the spire is due West. Draw a plan showing the positions of A, B, C, and P on a scale of 2 inches to the mile. Measure the angle BAP and the distance AP.

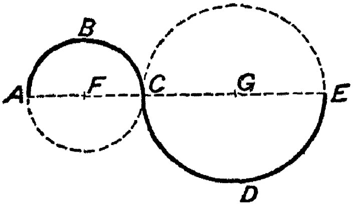

FIG. 323.

213. **Geometrical Patterns.**—The copying of geometrical patterns is an interesting and instructive exercise. Sufficient information about the figure must be given to enable the student to see the method of constructing the various parts of which the figure is composed.

Curves of various shapes can be made by combining arcs of circles. Wherever two arcs join into one *continuous* curve **the junction must be made on the line joining the centres of the two arcs.** Thus in Fig. 323 the two arcs ABC, CDE are drawn from centres F and G; they form one continuous curve, the junction occurring at C on the line joining their centres F and G.

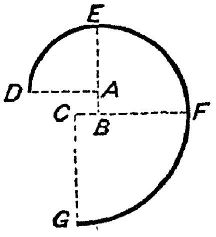

FIG. 324.

Similarly Fig. 324 shows a continuous curve made up of three circular arcs DE, EF, FG, whose centres are A, B, C respectively. The junction of the first pair of arcs occurs at E, on the line joining the centres A and B; and the junction of the second pair of arcs occurs at F on the line joining the centres B and C.

When two arcs form a continuous curve, they *form parts of circles which touch one another*, and the junction of the arcs is the point of contact of the complete circles. Thus in Fig. 323 the dotted arcs complete the circles, and *C* is the point of contact of the complete circles. This explains the rule for the junction of the arcs, as two circles can only *touch* on the line joining their centres. Tn.3.

Similarly if a straight line and an arc are joined so as to form a continuous line, the straight line must be a tangent to the arc at the point of junction ; *i.e.* the radius of the arc at the point of junction must be perpendicular to the straight line.

Example.—*Copy Fig. 325 to the given dimensions. A and B are the centres of all the arcs of which the figure is composed.*

Draw the line *CE*, measuring the lengths indicated on the figure.

A is the centre of the arc *DGC*, and *B* of the arc *CF*. The junction must be made at *C*, since *C* lies in the line *AB*.

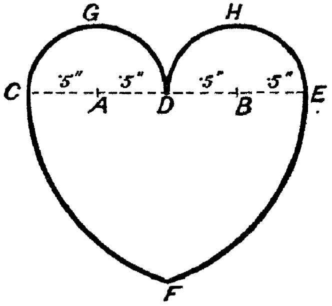

Fig. 325.

Hence we draw the semi-circle *DGC* with centre *A*, and then the arc *CF* with centre *B* and radius *BC*.

Similarly we draw the semi-circle *DHE* with centre *B*, and then the arc *EF* with centre *A* and radius *AE*.

The point *F* is determined by the intersection of the two arcs. The arcs *CF*, *FE* do *not* form a continuous curve because *F* does *not* lie on the line *AB*.

EXERCISES LXXVI (Practical).

1. Copy Fig. 326, making the side of each small square equal to 2 cms. Draw 25 circles, each of radius 1 cm., having their centres at the corners of the small squares. What is the greatest number of circles in contact with any one circle in the figure?

2. Copy Fig. 327, making the side of each small equilateral triangle equal to 2 cms. Draw 15 circles, each of radius 1 cm., having their centres at the corners of the small triangles. What is the greatest number of circles in contact with any one circle in the figure? Are these circles more closely or less closely "packed" than those in the preceding figure?

3. Copy Fig. 328, making the radius of each semicircle 1 cm.

4. Copy Fig. 329. Each diagonal of the rectilinear figure is to be 3″ long; each angle at the centre of the figure is 45°. The centre of each arc is at a vertex of the rectilinear figure, and the radius of each arc is half one side of the rectilinear figure.

5. Copy Fig. 330. Make the diagonal of the square 2″. All the arcs have their centres at the corners of the square, and the radius of the outer arcs is to exceed the radius of the inner arcs by ·2″.

6. Copy Fig. 331. Make the diameter of the outer circle 8 cms., and divide this diameter into four equal parts. Complete the figure by semicircles.

7. Copy Fig. 332. *HKLM* is to be a rhombus of side 2·7″, and shorter diagonal 2″. *H, K, L, M* are the centres of all the arcs of which the curves are composed. The radii of the arcs whose centres are *H* and *L* are to be ·6″ and ·9″.

8. Copy Fig. 333. The diameters of the two complete circles are to be 3 cms. and 2 cms. respectively. *C* is the centre of the two arcs indicated by the arrows. *AC* is to be 6 cms.; and the distance between the inner and outer ovals is to be 7 mm.

9. Copy Fig. 334, to the dimensions given. The figure supplies all the necessary data, if the principle of *symmetry* is properly applied. Obtain the lengths ¼″ and ⅛″ by dividing an inch into equal parts.

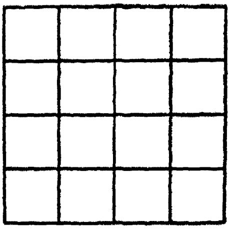

Fig. 326.

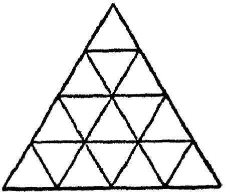

Fig. 327.

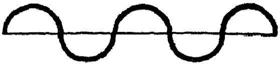

Fig. 328.

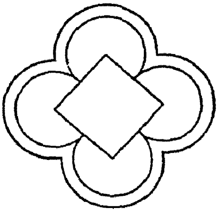

Fig. 330.

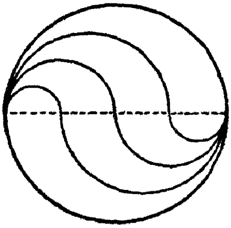

Fig. 331.

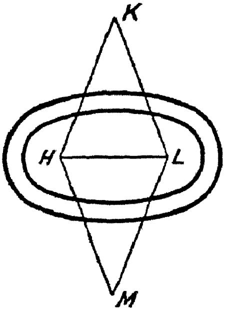

Fig. 332.

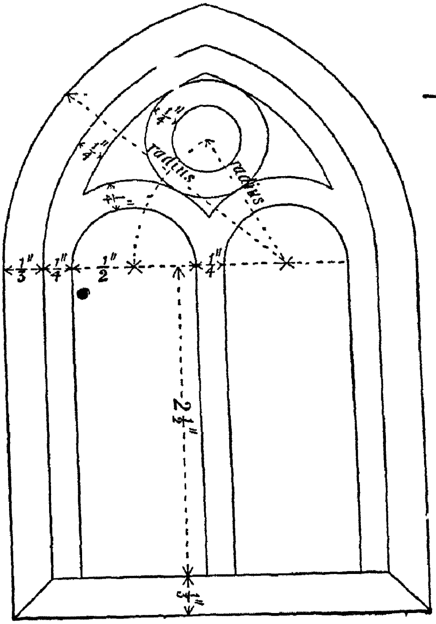

Fig. 334.

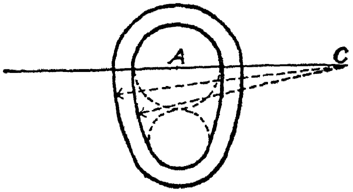

Fig. 333.

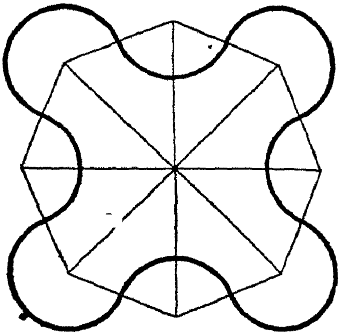

Fig. 329.

CHAPTER XIV.

MISCELLANEOUS PROPOSITIONS.

THIS Chapter deals with certain miscellaneous Propositions which are important in themselves and are also valuable as a training in the methods of solving Riders. The proofs in this Chapter are not always given in full.

CONSTRUCTION OF TRIANGLES, ETC.

214. PROBLEM 1.—**To construct a triangle given two angles and the perimeter.**[†]

Let P be the given perimeter (Fig. 335), X and Y the two given angles.

[*Analysis.*—Suppose that ABC is the required triangle, $\angle B = \angle X$, $\angle C = \angle Y$. Produce BC in both directions, making $BD = BA$ and $CE = CA$; then $DE = P$. It is easily proved (T.1, T.2) that $\angle ABC = 2\angle D$, $\angle ACB = 2\angle E$. The construction is now obvious.]

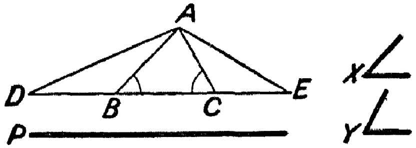

FIG. 335.

Cons.—Draw $\triangle ADE$ making $DE = P$, $\angle D = \frac{1}{2}\angle X$, $\angle E = \frac{1}{2}\angle Y$.
Draw AB, AC making $\angle DAB = \frac{1}{2}\angle X$, $\angle EAC = \frac{1}{2}\angle Y$.

[Prove by T.1, T.2*c*].

PROBLEM 2.—**To construct a quadrilateral given the two diagonals, the angle between them, and two opposite sides.**

Let P, Q be the two diagonals (Fig. 336), X the angle between them, and S, T the opposite sides.

[*Analysis.*—Suppose that $ABCD$ is the required quadrilateral, so that $AC = P$, $BD = Q$, $\angle AED = \angle X$, $AD = S$, $BC = T$. Draw AF, CG each equal and parallel to BD; then AF is equal and parallel to CG. Also $ABDF$, $CBDG$, $ACGF$ are parallelograms. Hence $AC = P$, $CG = Q$, $\angle ACG = \angle X$, $AD = S$, $DG = T$.]

[†] The perimeter of a triangle or polygon is the sum of its sides.

Cons.—Make $ACG = \angle X$, $CA = P$, $CG = Q$. With centres A, G, and radii S, T respectively draw arcs intersecting at D.
Draw DB equal and parallel to GC. Then $ABCD$ is the required quadrilateral.

[Prove by P.2*c*, and Pm.2.]

NOTE 1.—Fig. 336 is useful in other cases when two *opposite* sides of the quadrilateral are given.

NOTE 2.—The intersecting arcs will usually give two possible positions for D. Thus there are usually two solutions.

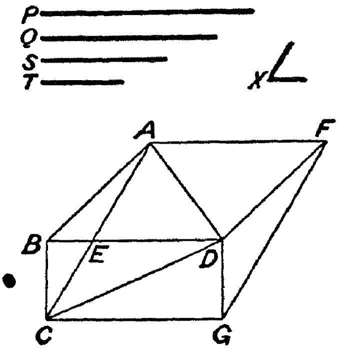

FIG. 336.

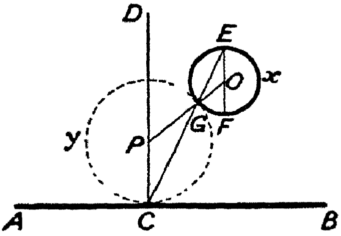

FIG. 337.

PROBLEM 3.—**To draw a circle to touch a given circle and to touch a given straight line at a given point.**
Let x be the given circle, O its centre, AB the given line and C the given point. (Fig. 337.)

Cons.—Draw $CD \perp AB$. Draw the diameter $EOF \parallel CD$.
Join EC, meeting $\bigcirc^{ce}$ of x at G. Join OG, meeting CD at P.
Describe a circle with centre P and radius PG.

[Prove $PG = PC$; etc.]

NOTE 1.—If in this construction F be used instead of E, a circle is obtained which touches AB at C and touches x internally.

NOTE 2.—There is another method of solving this problem which the student will easily discover for himself by the aid of the following hint:—
Draw two lines $\parallel AB$ (Fig. 337), one on each side of AB at a distance equal to the radius of x. Let CD meet these lines in H and K. Apply to this figure Method II. of § 212 (p. 291), and remember that there are two circles to be found.
This construction is rather more complicated than that given above, but will give more accurate results.

EXERCISES LXXVII (Riders).

1. Construct a right-angled triangle given the hypotenuse and the sum of the sides.

2. Construct a right-angled triangle given the hypotenuse and the difference of the sides.

3. Construct a triangle given the middle points of its sides.

4. In a triangle one acute angle is double or treble of another. In either case show that the triangle can be divided into two isosceles triangles.

5. Construct a triangle given the base, the vertical angle, and the point in which the bisector of the vertical angle meets the base.

6. Construct a quadrilateral given the two diagonals, the angle between them, and the angles which one of them makes with a pair of opposite sides.

7. Construct a quadrilateral given the two diagonals, the angle between them, and the angles at which each pair of opposite sides intersect.

8. Describe a circle to touch a given circle and two given tangents to that circle.

9. Describe a circle to touch three given equal circles (*a*) externally, (*b*) internally.

10. Describe a circle to intercept equal chords of given length on the three sides of a given triangle.

11. In a given rhombus inscribe two equal circles each touching two sides and the other circle.

12. In an equilateral triangle inscribe three equal circles, each touching two sides and the other two circles.

CONCURRENT LINES.

215. It is often required to prove that three or more lines are concurrent, *i. e.* meet one another at a single point. The following illustrations of this type of Rider should be carefully studied.

THEOREM 1.—**The three lines which bisect the sides of a triangle at right angles are concurrent.**

In Fig. 338 *ABC* is the triangle, and its sides *BC, CA, AB* are bisected at right angles by the lines *H, K, L* respectively.

Let K, L meet at O.
$\because$ K bisects CA at right angles, $\therefore$ $OC = OA$. L.3.
$\because$ L bisects AB at right angles, $\therefore$ $OB = OA$. L.3.
Thus $OC = OB$; whence O lies on the line H. L.3.
That is, the lines H, K, L are concurrent.

NOTE.—O is obviously the circum-centre. If ABC is an acute-angled triangle, O lies within ABC. If A is a right-angle, O lies on BC.

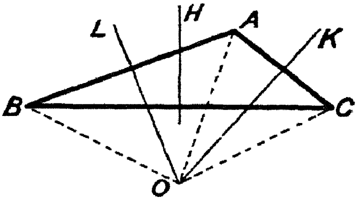

FIG. 338.

THEOREM 2.—**The three lines which bisect the angles of a triangle are concurrent.**

In Fig. 339 the lines H, K, L bisect the angles of the triangle ABC.

Let K, L meet at O. Draw OD, OE, OF $\perp$ the sides.

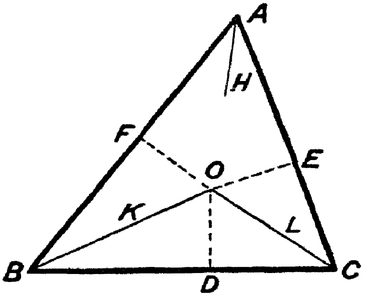

FIG. 339.

$\because$ K bisects $\angle B$, $\therefore$ $OF = OD$. L.4.
$\because$ L bisects $\angle C$, $\therefore$ $OE = OD$. L.4.
Thus $OF = OE$; whence O lies on L.4.
That is, the lines H, K, L are concurrent.

NOTE.—O is obviously the in-centre of the triangle ABC.

216. **The centroid.**—We will now show that the three medians of any triangle are concurrent. The point where the three medians intersect is called the **centroid** of the triangle.

THEOREM 3.—**In any triangle the three medians are concurrent.**

In △ ABC (Fig. 340) take E and F the middle points of the sides AC, AB. Let the medians BE, CF meet at G.

Required to show that AG produced bisects the side BC, and is therefore the third median.

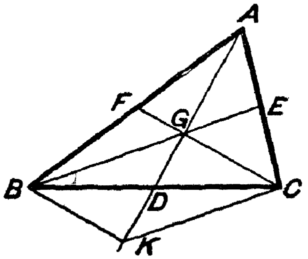

FIG. 340.

Cons.—Draw $CK \parallel EB$ meeting AG produced at K. Join BK.

Proof.—$\because AE = EC$ and $EG \parallel CK$, $\therefore AG = GK$. T.7.

$\because AG = GK$ and $AF = FB$, $\therefore FG \parallel BK$, *i.e.* $GC \parallel BK$. T.7.

$\therefore BGCK$ is $\parallel^{\text{m}}$ and its diagonals bisect each other. Pm.4.

$\therefore GK$, *i.e.* AG produced, bisects BC.

That is, the three medians are concurrent.

COROLLARY.—**The point of intersection of the medians** (*i.e. the centroid*) **is one of the points of trisection of each median.**

For $BG = CK$ (Pm. 1); but $CK = 2\ GE$ (T.7).

$\therefore BG = 2\ GE$.

Similarly $CG = 2\ GF$; also $AG = GK = 2\ GD$.

NOTE.—The following alternative proof (given in outline) is instructive.

The medians *BE, CF* are upposed to intersect in *G* and *AG* is joined and produced.

Since $AE = EC$, $\triangle AGE = \triangle CGE$ [Ar.5] and also $\triangle ABE = \triangle CBE$ [Ar.5]. By subtraction $\triangle ABG = \triangle CBG$. [This is the converse to the first Rider in § 145.]

Similarly, since $AF = FB$, $\triangle ACG = \triangle BCG$. Thus $\triangle ABG = \triangle ACG$. **Hence** *AG* bisects *BC* [Rider § 145].

EXERCISES LXXVIII (Riders).

1. Prove that the centroid of an equilateral triangle is the same as the circum-centre, and also as the in-centre.

2. Given the centroid and the base, construct the triangle.

3. Given the base of a triangle and the length of the median which bisects the base find the locus of the centroid.

4. In Fig. 340 prove that the six triangles (*AGF, AGE*, &c.) into which $\triangle ABC$ is divided by the medians are all equal in area.

5. In Fig. 340 if $AB > AC$ prove $EB > FC$. [Use T.6*c*, then T.6.]

6. Show how to construct a triangle given the lengths of its three medians. [Use $\triangle$ *GKC.*]

7. Show how to construct a triangle given the lengths of two of the medians and the angle between them.

8. Show how to construct a triangle given the base and the lengths of the two medians drawn through the ends of the base.

9. Show how to construct a triangle given one side and the length of any two medians.

10. Prove that the medians are concurrent by showing that the area of the triangle *AGD* is zero, where *D* bisects *BC*. [Prove that $\triangle$s *AGB, BGD* are together equal to half the triangle, *i.e.* to $\triangle ADB$.]

217. **Two Conditions determine a Straight Line.**—A straight line is completely determined by two conditions.

Thus the position of a straight line is determined if we know

(1) That it passes through *two* given points ; *or*

(2) That it passes through *one* given point and is parallel to *one* given straight line ; *or*

(3) That it passes through *one* given point and is perpendicular to *one* given straight line ; and so on.

Note that the two conditions resolve into either two points on the line, or one point on the line and the direction of the line.

A line is completely determined if we know that it bisects a given angle. (compare Theorem 2). This appears to be one condition only ; but is in reality two ; for it implies first that the line passes through the angular point, and second that its direction is midway between the directions of the two arms.

218. **Various Methods of Proof.**—Note the following account of the various methods of proof which are available for riders on the concurrence of three lines.

In Theorems 1 and 2 a *direct* method of proof was adopted; *i.e.* it was proved that H passed through the intersection of K and L. The direct method often involves the use of loci.

Riders of this type are more frequently solved by some *indirect* method.

Suppose that H, K, L are the three lines, that K and L intersect at G, and that H is determined by two given conditions. In the most common form of indirect proof we draw through G a line fulfilling one condition, and then prove that it also fulfils the other, and therefore is identical with H. (Compare Methods 1 and 2 below.)

It is also possible to draw through G two lines, each fulfilling one of the given conditions, and to show that these two lines are identical; this proves the existence of a line through G satisfying both the conditions which determine H, and therefore identical with H. (Method 3.)

Thus in attempting to solve Theorem 3 as a rider the following methods should suggest themselves (see Fig. 340):—

Indirect Methods of proving that AD passes through G.

(1) Prove that AG passes through D, the middle point of BC.
(2) Prove that DG passes through A.
(3) Prove that AG, GD form one straight line.

Direct Methods of proving that AD passes through G.

(4) Prove that area of $\triangle GAD$ is zero.
(5) Prove that the perpendicular from G on AD is zero.
(6) Prove that $\angle GAD$ is zero.
(7) Prove that $GA+GD=AD$.
(8) Regard the medians as loci.

Perhaps any of these methods could be used; but methods (1) and (4) give the only easy proofs.

219. **The Orthocentre.**—It can be proved that the three lines drawn from the vertices of a triangle at right angles to the opposite sides are concurrent. The point where these perpendiculars intersect is called the **orthocentre**

of the triangle; also the triangle whose vertices are the feet of these three perpendiculars is called the **pedal triangle.**

THEOREM 4.—**The three lines drawn from the vertices of a triangle perpendicular to the opposite sides are concurrent.**

Given that the perps. BE, CF intersect in P (Fig. 341),
Required to prove that $AP \perp BC$, *i.e.* that the three perpendiculars are concurrent.

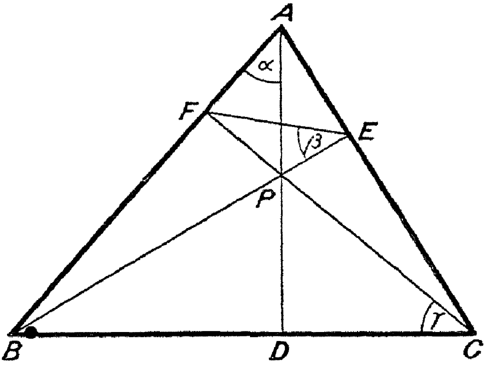

FIG. 341.

Proof.—$\because \angle AFP = \mathrm{R} = \angle AEP$; $\therefore \angle AFP + \angle AEP = 2\mathrm{R}$.
Hence quad. $AFPE$ is cyclic. AC.3*c*.

$\therefore \angle \alpha = \angle \beta$, being angles in the same segment on base FP. AC.2.

$\because \angle BFC = \mathrm{R} = \angle BEC$,
$\therefore$ the points B, F, E, C are concyclic. AC.2*c*.

$\therefore \angle \beta = \angle \gamma$, being angles in the same segment on base FB. AC.2.

Hence $\angle \alpha = \angle \gamma$.

In $\triangle$s ABD, CBF, $\because \angle \alpha = \angle \gamma$, $\angle B = \angle B$,
$\therefore$ the remaining angles ADB, CFB are equal. T.1.
But $\angle CFB = \mathrm{R}$ (*Hyp.*); hence $\angle ADB = \mathrm{R}$.
that is $AD \perp BC$.

NOTE 1.—In Fig. 341, where is the orthocentre of $\triangle PBC$? The perpendicular from P on BC is PD, from B on PC is BF, and from C on BP is CE. Thus the three perpendiculars of $\triangle PBC$ meet at A; thus A is the orthocentre.

Similarly the orthocentre of $\triangle PAB$ is C, and of $\triangle PCA$ is B. Thus, taking the four points A, B, C, P, any three of them are the vertices of a triangle whose orthocentre is the fourth.

Obviously the orthocentre of an obtuse-angled $\triangle$ falls outside it.

NOTE 2.—The above proof assumes that the triangle is acute-angled. To demonstrate the theorem for an obtuse-angled triangle, use the same figure (Fig. 341).

Let PBC be the given triangle. Draw $BF \perp CP$, and $CE \perp BP$; let BF and CE meet at A; join AP. Required to prove $AP \perp BC$. *The proof is exactly as given above.*

NOTE 3.—The student is recommended to carry out for himself a second method of proof by joining AP, drawing $PD \perp BC$ and proving that APD is a straight line. The proof depends on showing that $\angle APE = \angle BPD$ and using Theorem A. 2*c*.

NOTE 4.—Another valuable method for showing that three lines pass through a point is now open to us. For if we can show that the three lines are either the perpendiculars, or the medians, or the bisectors of the angles, or the perpendicular bisectors of the sides of some triangle in the figure it will follow by what has been already proved that they are concurrent.

EXERCISES LXXIX (Riders).

1. Where is the orthocentre of a right-angled triangle?

2. Show that the orthocentre of an equilateral triangle coincides with its centroid.

3. In Fig. 341 prove that the following quadrilaterals are cyclic:— $BFEC$, $CDFA$, $AEDB$, $AFPE$, $BDPF$, $CEPD$. AC.2*c*.

4. In Fig. 341 prove $\angle DFP = \angle EFP$ and deduce that *the orthocentre of a triangle is the in-centre of the pedal triangle.* AC.2.

5. In Fig. 341, if ED be produced to E' prove that $\angle E'DB = \angle FDB$, and deduce that the *angular points of a triangle are the ex-centres of the pedal triangle.* AC.2.

6. Given the base and vertical angle of a triangle find the locus of the orthocentre. L.5.

7. Given the base and the orthocentre of a triangle construct the triangle.

8. In Fig. 341 prove that the join of the middle points of AP and BC is perpendicular to FE. Ch.7.

9. In Fig. 341, if AD meet the circum-circle of the triangle ABC in Q, prove that $PD = DQ$. C.2; AC.2.

10. In Fig. 341 prove that the triangles AFE, BDF, CED are equiangular to one another, and to the triangle ABC. AC.3.

COLLINEAR POINTS.

220. It is often necessary to prove that three points lie on a straight line, *i. e.* are *collinear*. The method of doing this has been indirectly illustrated in the course of the previous paragraphs. Thus in Theorem 3 to show that AG, GD form one straight line is the same as to show that A, G, and D are collinear. So again in Theorem 4 we have really been investigating whether A, P and D are collinear, and this can be proved by showing that $\angle APE = \angle BPD$ [see Note 3]. We shall illustrate this type by the following important Rider.

THEOREM 5.—**ABC is a triangle and P is any point upon its circum-circle. PD, PE, PF are perpendicular to the sides BC, CA, AB respectively; prove that D, E, F are collinear.** (Fig. 342.)

[*Analysis.*—To prove that F, D, E in the figure are collinear we join FD, DE and endeavour to prove $\angle FDB = \angle EDC$ (using A. 2*c*). Since it is not immediately obvious that these angles are equal, we must try to discover other angles in the figure equal to them; in the hope that it may be easier to prove these angles equal to one another.

Join PB, PC. Now $\because$ $\angle BFP = \text{R} = \angle BDP$, $\therefore$ $BFDP$ is cyclic [AC.2*c*], $\therefore$ $\angle FDB = \angle FPB$ being angles in the same segment (on base FB). Similarly $\angle CDE = \angle CPE$. Hence $\angle FDB$ will be equal to CDE, if we can prove $\angle FPB = \angle CPE$. Now these two angles are angles in two right-angled triangles, and RT.1 shows that they are equal if we can prove $\angle FBP = \angle ECP$. But this follows at once by AC.3, Cor.

Hence we can arrange the proof as follows :]

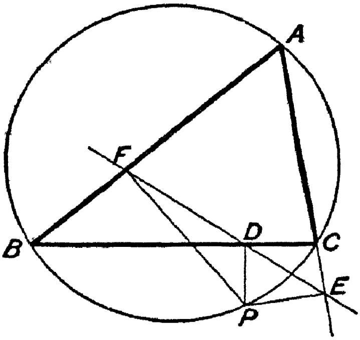

FIG. 342.

Proof (in outline): 1. $\angle ECP = \angle FBP$. [AC.3.].
2. $\therefore$ $\angle CPE = \angle FPB$. [RT.1.]
3. $\therefore$ $\angle FDB = \angle EDC$. [AC.2.]
4. $\therefore$ FDE is a straight line. [A.2*c*.]

The line FDE is called the **Pedal Line** *or, sometimes, the* **Wallace Line** *of P with respect to the triangle ABC.*

EXERCISES LXXX (Riders).

1. If P be any point from which perpendiculars PD, PE_1 PF are drawn to the sides BC, CA, AB of a triangle prove that if DEF is a straight line P lies on the circum-circle of ABC.

2. Show how to find a point such that the feet of the perpendiculars from it upon the sides of a quadrilateral are all collinear.

3. $ABCD$ is a cyclic quadrilateral. Circles are described on AB, AC, AD as diameters. Prove that these circles intersect in three points (beside the point A) which are collinear.

4.* If P and Q be any two points on the circum-circle of ABC prove that the angle between the pedal lines of P and Q is equal or supplementary to the angle PAQ. T.1 ; AC.2.

5.* If P and Q be at opposite ends of a diameter of the circum-circle of ABC prove that the pedal lines of P and Q are perpendicular to each other. Use 4.

6.* ABC and LMN are two equilateral triangles inscribed in the same circle. Prove that the pedal lines of L, M, N with regard to the $\triangle ABC$ form an equilateral triangle by their intersections. Use 4.

ORTHOGONAL CIRCLES.

221. DEFINITION.—**The angle between two curves which meet at a point is the angle between the tangents to the two curves at that point.**

Thus in Fig. 343 the angle between the two arcs BEA and BHA at the point A is the angle CAD, and the angle between the same two arcs at the point B is the angle CBD.

THEOREM 7.—**If two circles cut one another they form the same angle at each intersection.**

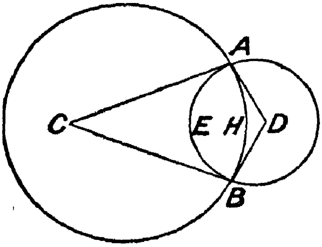

FIG. 343.

For in Fig. 343 $CA = CB$, and $DA = DB$. TH.2.

Hence $\triangle CAD \equiv \triangle CBD$, and $\angle CAD = \angle CBD$. C.3.

DEFINITION.—**Two circles are said to cut *orthogonally* when the two tangents at either point of intersection are at right angles.**

Thus in Fig. 343 the two circles cut orthogonally provided that $\angle CAD = \angle CBD = R$.

EXERCISES LXXXI (*Riders*).

1. If two circles cut orthogonally the tangents to either circle at the points of intersection meet at the centre of the other circle.

2. If two circles cut orthogonally the square on the distance between their centres is equal to the sum of the squares on their radii; and conversely.

3. Draw a circle with a given centre to cut a given circle orthogonally.

4. Three equal circles are drawn with their centres at the vertices of an equilateral triangle. Draw a circle to cut all three circles orthogonally.

MAXIMA AND MINIMA.

222. The **maximum** value of a quantity is in Geometry usually the greatest value which it can possibly have subject to the conditions which define it, and the **minimum** value is the smallest.

These are not the strict mathematical meanings of the words maximum and minimum. In the case of the tides, when the tide is "coming in" the water gradually rises up a pier until the moment of "high tide" when it begins to fall. The height of the water at high tide is a "maximum" although the height on the particular day in question may be actually less than the height on the following day.

The characteristic of a mathematical "maximum" is not so much that it should be the greatest possible value as that it should be greater than the values which immediately precede or follow it. Similarly for the "minimum." In Elementary Geometry, however, this point is not of much importance.

We have already had several examples of the maximum and minimum value of geometrical quantities. Thus in Prop. T.5 it was proved that the *minimum distance of a point P outside a line from a point in the line is the length of the perpendicular from P to the line.*

Again we have seen (Ques. 3, Ex. *LXIV*, p. 247) that if P be a point inside a circle and C be the centre of the circle the maximum distance of P from a point on the circle is found by producing PC to meet the circle, and the minimum distance by producing CP to meet the circle. We shall investigate some other important cases of maxima and minima.

PROBLEM 4.—**Given two points A and B on the same side of a line X to find a point P in the line such that the sum of the lines AP, BP should be a minimum** (Fig. 344).

[*Analysis.*—Let B' be the image of B in the line X, *i. e.* a point such that BB' is perpendicular to the line and $BN = NB'$; then if P' be a point on the line we have proved (L.3) that $BP' = B'P'$, whence $AP' + P'B = AP' + P'B'$. Thus the problem reduces itself to finding a point P in X such that $AP + PB'$ should be a minimum. This is clearly the point where AB' meets the line X.]

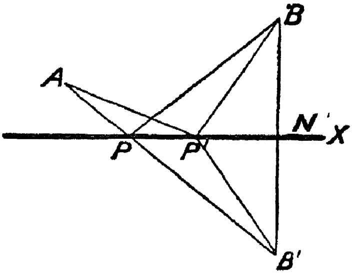

FIG. 344.

Cons.—Draw $BN \perp$ to the line X and produce to B' making $NB' = NB$. Join AB' meeting the line in P. Then P shall be the point required. Take any other point P' in the line. Join AP', $P'B'$, $P'B$.

Proof.—$\because$ X bisects BB' at right angles, *Cons.*

$\therefore P'B' = P'B$, and $PB' = PB$. L.3.

Thus $AP' + P'B = AP' + P'B' > AB'$ T.4.

$= AP + PB'$

$= AP + PB$

i. e. $AP' + P'B > AP + PB$.

Hence P is the point required, for the value of $AP + PB$ is smaller than the value $AP' + P'B$, wherever P' falls on the line X.

NOTE 1.—It is easy to prove that AP, BP are equally inclined to X. Conversely it is easy to prove that if AP, BP are equally inclined to X, then $AP + PB$ is a minimum.

NOTE 2.—The student who is acquainted with the laws of optics will see that if a ray of light emanating from A is incident on a mirror X at P it will be reflected along PB. Thus a ray of light proceeding from A to B by reflection at the mirror X takes the *quickest possible path*. This is a general law in optics of the greatest importance.

223. THEOREM 7.—**Of all triangles of given area upon a given base the isosceles triangle is the one of minimum perimeter.**

Given a series of triangles upon base AB (Fig 345), and having the same area, *i. e.* (Ar.5) a series of triangles upon base AB whose vertices lie on a line CD parallel to AB;

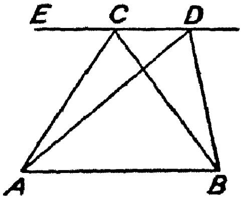

FIG. 345.

Required to prove that of these triangles the isosceles triangle has the smallest perimeter.

Proof.—Let ACB be the isosceles triangle having AC equal to BC, and let ADB be any other triangle having an equal area.

Then $\angle CAB = \angle CBA$. T.2.

$\therefore \angle ACE = \angle BCD$. P.1*c*.

Hence by the preceding rider,

$$AC + CB < AD + DB;$$

$$\therefore AC + CB + BA < AD + DB + BA.$$

Hence the perimeter of the isosceles triangle ACB is a minimum.

THEOREM 8*.—**Of all triangles of given area the equilateral triangle has the minimum perimeter.**

(1) † There *must be* a triangle, or a set of triangles with the given area having a minimum perimeter. No other hypothesis is thinkable.

(2) In Fig. 345 let ADB be a triangle having the given area. Then if $AD \neq DB$ it follows from Theorem 7 that a triangle ACB can be found having the same area and a less perimeter. Thus *any* triangle which is *not* equilateral can be reduced to a triangle of equal area and *less* perimeter.

Hence *the triangle or triangles of minimum perimeter must be equilateral.*

(3) † It is easy to show by superposition that two equilateral triangles having unequal perimeters are unequal in area. That is, all equilateral triangles having the given area have the same perimeter, and *this perimeter must therefore be the least possible.*

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† Steps (1) and (3) are often omitted, but are necessary for logical completeness.

THEOREM 9.*—**Of all quadrilaterals of given area the square has the minimum perimeter.**

(1) There *must be* a quadrilateral or a set of quadrilaterals having a minimum perimeter.

(2) If *any* two adjacent sides of a quadrilateral are unequal, it is easy by using Theorem 7 to replace it by an equal quadrilateral with a less perimeter.

Hence the quadrilateral of minimum perimeter must have all its sides equal, *i. e.* it *must be a rhombus.*†

(3) If the rhombus is not a square it is easy to see that a rectangle on the same base and of the same altitude is equal in area and of less perimeter.

Hence the quadrilateral of minimum perimeter *must be a square.*

(4) It is easy to show that two squares of unequal perimeter are unequal in area. That is, all squares having the given area have the same perimeter, *which must therefore be the minimum perimeter.*

224. PROBLEM 5.—**Through either point of intersection of two circles draw a chord terminated by the two circles and such that its length shall be a maximum.**

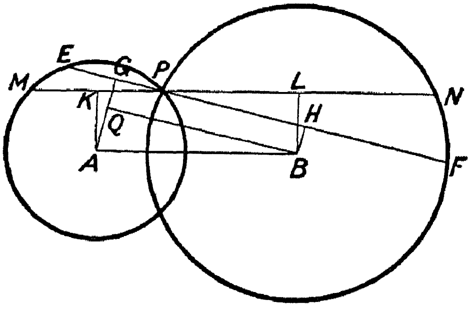

FIG. 346.

Given that A, B are the centres of two circles intersecting in P (Fig. 346),

Required to draw through P a chord MPN such that MN is a maximum.

Cons.—Draw $MPN \parallel AB$. Then MN shall be the chord required. Take any other chord EPF. Draw $AK, BL, AG, BH, BQ \perp$ to MP, PN, EP, PF, AG respectively.

Proof.	$KP=\frac{1}{2}MP$, and $PL=\frac{1}{2}PN.$	Ch.1.
	$\therefore KL=\frac{1}{2}MN$; whence $AB=\frac{1}{2}MN.$	Pm.1, etc.
Similarly	$GH=\frac{1}{2}EF$; whence $QB=\frac{1}{2}EF.$	
But	$AB>QB.$	T.3.
Hence	$MN>EF.$	

---

† There *can* be two rhombi having the given area and *different* perimeters. Hence the problem is not yet solved.

225. **Direct and Indirect Methods.**—The preceding examples illustrate the two methods of solving riders on maxima and minima, viz. :—

(1) *The Direct Method* in which we prove that some particular value of the varying magnitude is greater (or less) than all other possible values. See Problems 4 and 5.

(2) *The Indirect Method* in which we start by assuming that there must be a maximum (or minimum) value of the varying magnitude, and then show that no possible figure *except one* can give the maximum or minimum value of the varying magnitude. See Theorems 8 and 9.

EXERCISES LXXXII (Riders).

1. If A and B be on opposite sides of a straight line find a point P in the line such that $AP+BP$ is a minimum.

2. If A and B be two points (1) on the same side, (2) on opposite sides of a straight line, find a point P in the line such that $AP-BP$ is a maximum.

3. Given two of the sides of a triangle prove that the area is a maximum when these sides contain a right angle.

4. Prove that of all triangles with given base and given vertical angle the isosceles triangle is the one of maximum area.

5. P and Q are two points and AB, CD are two parallel lines lying between the points, P and CD being on opposite sides of AB. Find points M and N in AB and CD respectively, such that while MN is fixed in direction $PM+MN+NQ$ is a minimum.

[Draw $QT \parallel$ the given direction of MN and equal to the constant length of MN. Join PT meeting AB in M. Draw $MN \parallel$ given direction, meeting CD in N. Then M and N are points required.]

6. Through a point P lying in the angle AOB draw a line APB cutting off from AOB a triangle of minimum area.

[The line APB is bisected in P.]

7.* Of all equilateral triangles whose sides pass through three given points P, Q, R show how to construct the triangle of maximum perimeter. § 224.

8.* OP, OQ are two intersecting straight lines, and A and B are two points in the angle POQ. Find points X and Y in the lines OP, OQ such that $AX+XY+YB$ shall be a minimum.

9*. Of all triangles on a given base and of given perimeter the isosceles triangle has the greatest area.

226. We shall conclude this chapter with a few Riders, the solutions of which are briefly indicated. The student is recommended in every case to draw the figure, and to work the rider carefully for himself.

1. *Divide an arc of a circle into two parts, such that the sum of the chords of these parts is equal to a given line.*

> [Let AB be the chord of the given arc ACB. On AB describe a segment of a circle ADB containing an angle equal to $\frac{1}{2}\angle ACB$. With B as centre, and radius equal to the given sum of chords describe a circle cutting ADB in D. Join DB meeting the arc ACB in C. Then C is the required point of division.]

2. *OA, OB are two intersecting straight lines, and P is a point in the angle which they form. Draw through P a straight line terminated by OA, OB and such that P is one of its points of trisection.*

> [Draw $PN \parallel$ to OA meeting OB in N. Measure off NM, MB each equal to ON. Join BP producing to meet OA in A. Then BA is the line required. Draw $MQ \parallel$ to OA and prove by P.5.]

3. *Through a given point draw a straight line such that of the perpendiculars drawn to it from two given points one shall be twice the other.*

> [Let A, B, C be the three given points. Join BC. Trisect it in M, N. Join AM. Then AM shall be the line required. For proof draw perpendiculars from C and B on AM, and from N on the perpendicular from B. Use C.2. AN is a second solution. If CB be produced both ways to D and E so that $DB=BC=CE$, then AD, AE are two other solutions.]

4. *P is a point inside a quadrilateral $ABCD$ such that the area $PBAD$ = area $PBCD$; find the locus of P.*

> [Join BD. Suppose P and C on the same side of BD so that the quadrilateral $PBCD$ is re-entrant. Then area $PBAD = \triangle ABD + \triangle PBD$ and $PBCD = \triangle BCD - \triangle PBD$. $\therefore \triangle ABD + \triangle PBD = \triangle BCD - \triangle PBD$. $\therefore 2\triangle PBD = \triangle BCD - \triangle ABD$ = a constant quantity. Since $\triangle PBD$ is on a fixed base BD, and has a fixed area, $\therefore$ the locus of P is a straight line $\parallel$ BD. Also the middle point of AC is obviously one point on the locus.]

5. *Prove that the sum of the three medians of a triangle is less than the sum of the sides of the triangle.*

[Let AD, BE, CF be the medians of $\triangle\ ABC$. Produce AD to A' making $DA'=AD$. Join $A'C$. Prove $\triangle$s $A'DC$, ADB congruent by C.1. $\therefore\ A'C=AB$. But $A'C+CA>AA'$. $\therefore\ AB+AC>2\ AD$. Similarly $BC+BA>2\ BE$, $CA+CB>2\ CF$. Add these results. The student should also deduce this result by applying T.4 to $\triangle\ ADE$ in the figure.]

6.* *Find the locus of a point such that the difference of its perpendicular distances from two intersecting lines is a constant quantity.*

[Let OM, ON be the two intersecting lines. Draw OA at right angles to OM and on the same side of it as ON. Make OA equal to the constant difference of the perpendiculars. Draw ABC parallel to OM cutting ON in B. Bisect $\angle NBC$ by BP. This line is the locus. Take P any point on BP. Draw $PN \perp$ to ON, and $PM \perp$ to OM cutting AB in C. $PN=PC$ (L.4). $\therefore\ PM-PN=PM-PC=CM=AO=$ the constant quantity. If OD is drawn at right angles to ON on the same side as OM, and made equal to the constant difference, and if through D the line DEF is drawn parallel to ON cutting OM in E, the student can show that the bisector of $\angle FEM$ is also part of the locus. The advanced student should satisfy himself that if PB is produced beyond B the part thus produced also forms part of the locus, adopting the usual convention of signs.]

7.* *If $ABCD$ be a quadrilateral, and E, F, G, H, K, L be the middle points of AB, BC, CD, DA, BD, AC respectively, prove that EG, HF, KL are concurrent and bisect each other.*

[Join EF, FG, GH, HE, HK, KF, FL, LH. Since H, E are middle points of AD, AB, $\therefore\ HE \parallel$ to BD and equal to $\frac{1}{2}\ BD$. Similarly $FG \parallel BD$ and equal to $\frac{1}{2}\ BD$. $\therefore\ EH$, FG are equal and $\parallel$, $\therefore\ EFGH$ is $\parallel^{\text{m}}$ and its diagonals bisect each other; *i.e.* EG, HF bisect each other. Similarly HK, LF are both $\parallel$ to AB and equal to $\frac{1}{2}\ AB$, $\therefore$ they are equal and $\parallel$ to each other; $\therefore\ HKFL$ is $\parallel^{\text{m}}$, $\therefore\ HF$, KL bisect each other. $\therefore\ EG$, HF, KL bisect each other.]

8.* *To inscribe a square in a given triangle.*

[ABC is the triangle; draw AD perpendicular to BC; produce BC to E, making CE equal to BD; bisect the angle ADE by DF, meeting AE in F; through F draw FKL parallel to BC meeting AB, AC, AD in L, K, H respectively; draw KM, LN perpendicular to BC; $KMNL$ is the square required. Prove FH equal to HD, and FK equal to HL. To prove the latter, join FC, HB, and prove by Ar.5 that, if $FK \neq HL$, then $\triangle ACE \neq \triangle ABD$; which is absurd. This construction assumes that the angles at the base are acute.]

EXERCISES LXXXIII.

MISCELLANEOUS RIDERS.

ON TRIANGLES.

1. If in any triangle a line be drawn from either extremity of the base perpendicular to the bisector of the vertical angle, it will make with the base an angle equal to half the difference of the base angles, and it will make with the side an angle equal to half the sum of the base angles.

2. In Question 1 the line joining the middle point of the base to the foot of the perpendicular will be equal to half the difference of the sides of the triangle.

3. If *ABC* be a triangle whose sides *AB, AC* are produced to *N* and *M* prove by the method of § 215 that the bisectors of the angles *CBN, BCM, BAC* are concurrent.

4. Any straight line which is drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other two sides of the triangle.

5. The three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.

6. By drawing through *A, B, C* parallels to *BC, CA, AB* respectively forming a triangle *LMN*, prove that the perpendiculars from *A, B, C* to *BC, CA, AB* are concurrent.

7. Prove that the sum of any two medians of a triangle is greater than the third median.

8. Prove that the sum of the three medians of a triangle is greater than ¾ of the sum of the sides.

9. *ABC* is a triangle of which *AD, BE, CF* are the medians. *AG, BG* are parallel to *BE, AC* respectively. Show that *G, F, E* lie on a straight line, that *GD* is parallel to *CF,* and that *GC* bisects *FD.*

10. In any triangle the difference of the base angles is double the angle between the perpendicular to the base and the bisector of the vertical angle.

11.* In the triangle *ABC, AX* is the median, *AP* is the perpendicular to *BC,* and *E* and *F* are the middle points of *AC* and *AB* respectively; prove that the angles *FPE* and *FXE* are each of them equal to *BAC.*

12.* If the base of an isosceles triangle be produced to any point, the difference of the perpendiculars drawn from this point to the equal sides is constant.

13.* The sum of the perpendiculars from any point within an equilateral triangle to the three sides is constant.

14 *ABC* is any triangle, and lines *AD, BE, CF* are drawn, making the angles *BAD, CBE, ACF* all equal to one another; prove that these lines either meet in a point, or else form a triangle equiangular to the triangle *ABC.*

15.* Construct a right-angled triangle given the angles and the difference between the hypotenuse and one of the sides.

16.* Construct a right-angled triangle given one of the sides containing the right angle and the difference between the hypotenuse and the other side.

17.* Construct a triangle given the base, the difference of the sides, and the difference of the base angles.

18.* *If P be a point inside a triangle ABC prove that* $AB + AC > PB + PC$. Deduce that $PA + PB + PC$ is less than the sum of the sides of the triangle but greater than half their sum.

19.* The line joining any corner of a triangle to its ortho-centre is double the distance of its circum-centre from the side opposite that corner.

ON QUADRILATERALS AND POLYGONS.

20. Prove that the angle between the bisectors of two adjacent angles of a quadrilateral is half the sum of the two remaining angles of the quadrilateral.

21. A quadrilateral has two sides parallel and two sides equal but not parallel; show that the diagonals of the quadrilateral are equal.

22. If two opposite sides of a parallelogram be bisected and two straight lines be drawn from the points of bisection to two opposite angles, the two lines trisect the diagonal which passes through the other two angular points of the parallelogram.

23. *ABCDE* is a regular pentagon. Prove that the lines *AC, AD* trisect the angle *BAE;* and that *ACD* is an isosceles triangle having each of the angles *ACD* and *ADC* double the angle *CAD.*

24. The perimeter of a quadrilateral figure is greater than the sum of the diagonals, but less than twice the sum of the diagonals.

25. If a straight line be drawn in any direction through one angle of a parallelogram, the perpendicular to it from the opposite angle of the parallelogram is equal to the sum or difference of the perpendiculars to it from the two remaining angles of the parallelogram. Distinguish the cases of the sum and difference of the perpendiculars.

26. The quadrilaterals formed by either the four internal or the four external bisectors of the angles of any quadrilateral have their opposite angles together equal to two right angles.

27. The straight line, which joins the middle points of the oblique sides of a trapezium, is equal to half the sum of the parallel sides; and the part of this line intercepted between the diagonals of the trapezium is equal to half the difference of the parallel sides.

28. Show how to cut off the corners of an equilateral triangle so as to leave a regular hexagon.

29.* Show how to cut off the corners of a square so as to leave a regular octagon.

30.* The diagonals of the rectangle formed by the bisectors of the angles of a parallelogram are parallel to the sides of the parallelogram and are equal to the difference between them.

ON AREAS.

31. The figure formed by joining the middle points of the sides of a quadrilateral is a parallelogram and its area is half that of the quadrilateral.

32. Of all the parallelograms which can be formed with diameters of given lengths, the rhombus is the greatest.

33. The area of a quadrilateral is equal to the area of a triangle, two of whose sides are equal to the diagonals of the quadrilateral, the angle included by these sides being also equal to the angle between the diagonals.

34. Construct a triangle whose angles shall be equal to those of a given triangle and whose area shall be four times the area of the given triangle.

35. Divide a given straight line into two parts so that the square on one part may be double the square on the other part.

36. Divide a straight line into two parts, the difference of whose squares shall be equal to a given square.

37.* If any parallelograms $ABDE$, $ACFG$ be described externally on the sides AB, AC of any triangle ABC, and if DE, FG, or these lines produced, meet in H; then the parallelogram $BKLC$, described on BC and having its sides BK, CL equal and parallel to HA, will be equal to the sum of the parallelograms AD and AF.

38.* $ABCD$ is a parallelogram and P is any point; show that the sum or difference of the areas of the triangles PAB, PAD is equal to the area of the triangle PAC. Distinguish the cases for the sum and difference.

39.* If two chords of a circle intersect at right angles, the sum of the squares on the four portions of chords is equal to the square on the diameter of the circle.

40.* If a point P can be found inside a quadrilateral $ABCD$ such that the areas PAB, PBC, PCD, PDA are all equal, the quadrilateral is a parallelogram and the point is the intersection of the diagonals.

ON CIRCLES—CHORDS AND TANGENTS.

41. *AB* is a fixed chord in a circle. Show that the sum or difference of the perpendiculars on *AB* from the extremities of any diameter of the circle is constant.

42. In any segment of a circle less than a semicircle, if straight lines be drawn from points in the arc perpendicular to the chord, the greatest perpendicular will bisect the chord, and of any others, that which is nearer to the greatest is always greater than one more remote.

43. The inscribed circle of the triangle *ABC* touches the side *BC* at *D*. Prove that *AB* + *CD* is equal to half the perimeter of the triangle.

44. An escribed circle of the triangle *ABC* touches the side *BC* at *D*, and the other two sides *produced* at *E* and *F*. Prove that *AE* is equal to half the perimeter of the triangle.

45. In any triangle the difference of two sides equals the difference of the segments into which the third side is divided at the point of contact of the in-circle.

46. If a hexagon be circumscribed about a circle, the sums of the alternate sides shall be equal.

47. *AB* is a chord of a circle; show how to draw from *A* a straight line so that the part of it intercepted between *A* and the tangent at *B* may be bisected by the circumference of the circle.

48. *CD* is a chord of a circle perpendicular to the diameter *AB*, and *E* is any point in *AB*; if the lines *CE* and *DE* meet the circumference in *F* and *G* respectively, prove that *CF* is equal to *DG*, and that *FG* is perpendicular to *AB*.

49. If the line joining the in-centre and the circum-centre of a triangle pass through an angular point of a triangle, the sides terminating in that point are equal.

50. If the in-centre and circum-centre of a triangle coincide the triangle is equilateral.

51. The diagonals of a quadrilateral intersect at *O*; show that the centres of the circum-circles of the four triangles *AOB*, *BOC*, *COD*, *DOA* are at the angular points of a parallelogram.

52. If a circle be inscribed in a right-angled isosceles triangle the distance of the centre of the circle from the right-angle is equal to the difference between the hypotenuse and one of the equal sides.

53. The triangle, whose vertices are the three points of contact of the inscribed circle with the sides of a triangle, is always acute-angled. Also the triangle, whose vertices are the three points of contact of an escribed circle with the sides of a triangle, is always obtuse-angled.

54. Two circles whose centres are *O* and *O′* intersect at *P*, and through *P* are drawn *APB*, *CPD*, lines equally inclined to the line *OO′*: show that *AB* = *CD*.

55.* The line joining the in-centre and circum-centre of a triangle subtends at any angular point of the triangle an angle which is half the difference of the angles at the other angular points.

56.* Four common tangents are drawn to two circles which are external to one another; show that the two direct and the two transverse tangents intersect on the line joining the centres of the circles.

57.* The direct common tangents to two circles having external contact touch the circle described on the line joining the centres as diameter.

58.* Two equal circles are so placed that the tangent to either from the centre of the other is equal to a diameter; show that their transverse common tangents are each equal to a radius.

59.* If three circles touch one another externally in pairs, show that the tangents drawn to them at the three points of contact meet in a point.

60.* Find the magnitude of the angle subtended at the centre of a circle by the part of any tangent intercepted by the square circumscribing the circle.

ON CIRCLES—ANGULAR PROPERTIES.

61. The sums of the alternate angles of any polygon of an even number of sides inscribed in a circle are equal.

62. ABC is any triangle, X, Y, Z are the middle points of its sides, and AD is perpendicular to BC; prove that X, Y, Z, D are concyclic.

63. Prove that in any triangle the middle points of the sides and the feet of the perpendiculars let fall from the vertices on the opposite sides are concyclic.

64. AB and CD are two chords which intersect within the circle at E; prove that the angle AED is equal to the sum of the angles in the segments cut off by AD and BC.

65. If I is the in-centre of a triangle ABC, and I_1 the centre of the escribed circle which touches BC, show that IBI_1C are concyclic.

66. If O be the circum-centre of a triangle ABC prove that the bisector of the angle A and the line through O perpendicular to BC meet on the circum-circle.

67. If on the sides of a triangle as chords circles are described, such that the sum of the angles in their segments remote from the triangle is equal to two right angles, these circles will have one point in common.

68. If two chords of a circle intersect at right angles, the opposite arcs are together equal to a semicircle.

69. Circles are described on the sides of a quadrilateral as diameters; prove that the common chord of two circles on any two adjacent sides is parallel to the common chord of the other two circles.

70. On BC, CA, AB, the sides of a triangle ABC, any points A', B', C' are taken; show that the circles described about the triangles $AB'C'$, and $BA'C'$ and $A'B'C$ meet in a point.

71. Prove that the four straight lines bisecting the angles of any quadrilateral form a cyclic quadrilateral.

72. If four circles be described, all outside or all inside any quadrilateral so that each of them touches three of its sides, then their centres are concyclic.

73.* A, B, C, D are four points on a circle; show that the four points, where the perpendiculars from any point to the straight lines AB, CD meet AC, BD, are concyclic.

74.* From a fixed point O on a given circle there are drawn two chords OX, OY, equally inclined to a fixed chord; show that XY is fixed in direction.

75.* $ABCD$ is a quadrilateral inscribed in a circle. If AC and BD intersect at right angles at O, and if OQ be drawn perpendicular at AB, prove that QO produced will bisect CD.

76.* In a cyclic quadrilateral whose diagonals are at right angles, prove that the distance of the centre of the circum-circle from any side is half the opposite side.

77.* If two circles touch one another internally, and if a chord of the outer touches the inner circle, then the segments of this chord made by its point of contact subtend equal angles at the point of contact of the circles.

78.* AB and CD are perpendicular diameters of a given circle, and P is any point on the circumference; if the lines AP and BP and the tangent at P cut CD or CD produced in E, F and G, prove that EG is equal to FG.

79.* If from any point on a given arc of a circle perpendiculars are drawn to the radii bounding that arc, the distance between the feet of these perpendiculars is constant.

80.* A series of circles touch a fixed straight line at a fixed point; show that the tangents to them at the points where they are cut by a fixed parallel straight line all touch a fixed circle.

81.* $AICB$ is a diameter of a circle whose centre is I; DCE is any chord through C on a constant arc DE; if AD and BE meet in O, prove that the angle DOE is constant.

82.* If from any point in the circumference of a circle three lines are drawn to the angles of an inscribed equilateral triangle, one of these lines is equal to the sum of the other two.

83.* *AB, AC* are any two chords of a circle; if *P, Q* are the middle points of the minor arcs cut off by them, and if *PQ* cuts *AB, AC* in *X, Y,* show that *AX* is equal to *AY.*

84.* Two circles intersect at *A* and *B;* show that, if from any point *P* in the circumference of one circle, whose centre is *O, PAQ* and *PBR* are drawn to meet the other circle in *Q* and *R,* then *OP* and *QR* intersect at right angles.

85.* Two circles intersect, and through one of the points of section two straight lines are drawn to meet the circumferences in four points; show that if the two points on each circle are joined, the joining lines intersect at a constant angle.

86.* *A, B, C* are three points on a given circle; if the tangents at *A* and *B* meet in *O,* and from *O* a straight line be drawn parallel to *CA,* it meets *BC* in the diameter perpendicular to *CA.*

87.* The circles circumscribing the four triangles formed by four intersecting straight lines, go through a point which is concyclic with their centres.

ON CIRCLES—PROBLEMS.

88. Describe a circle of given radius to touch two given straight lines.

89. Describe a circle of given radius to touch two given circles.

90. Describe two circles of given radii to touch one another and a given straight line, on the same side of it.

91. Describe a circle to pass through a given point and to touch a given straight line at a given point.

92. Describe a circle to pass through a given point and to touch a given circle at a given point.

93. Describe a circle to touch two concentric circles and pass through a given point. When is this impossible?

94. Describe a circle to touch a given straight line and a given circle at a given point.

95. Describe a circle to touch a given circle, to have its centre in a given straight line and pass through a given point in that straight line.

96. Through a given point draw a straight line to make equal chords in two given equal circles. When will there be two solutions, one solution, or no solution possible?

97. Find a point *O* within or without any triangle *ABC* such that the circles described about *OAB, OBC, OCA* may all be equal.

98. Draw a tangent to a given circle such that the part of it intercepted by another given circle may be equal to a given line.

99. With the vertices of a given triangle as centres describe three circles which touch one another externally in pairs.

100. With the vertices of a given triangle as centres describe three circles which touch one another in pairs, so that two of the circles lie within the third.

101.* Draw a circle touching each of three given equal circles internally.

102.* Divide an arc of a circle into two parts such that the difference of the chords of these parts is equal to a given line.

ON LOCI.

103. If P and Q be two fixed points, and R move so that the perpendicular from P on QR bisects QR, the locus of R is a circle.

104. The locus of the middle points of all lines drawn from a fixed point to meet a fixed circle, is a circle.

105. Find the locus of the middle point of a straight line drawn from a given point to meet a given straight line of unlimited length.

106. Find the locus of the point P which moves so that the sum of the areas of the two triangles PAB, PCD is constant.

107. Find the locus of the centres of the circles inscribed in all right angled triangles standing on the same hypotenuse as base.

108. Given the base of a triangle, and the magnitude of its vertical angle, find the locus of the ex-centre which is opposite to the vertical angle.

109.* On a given base BC a triangle ABC is described, such that AC is equal to the length of the perpendicular from B on AC; find the locus of the vertex A.

110.* Find the locus of a point which is such that the sum of its perpendicular distances (1) from two parallel straight lines, (2) from two intersecting straight lines is a constant length.

111.* Find the locus of a point, such that the sum of the squares on its distances from three given fixed points is constant.

112.* PQ, a straight rod of given length, slides between two given rulers OP, OQ; PX, QX are perpendicular to OP, OQ; show that the locus of X is a circle.

113.* Given the vertical angle of a triangle and the sum of the sides containing it, find the locus of the circum-centre.

ON MAXIMA AND MINIMA.

114. Given two adjacent sides, describe a parallelogram of maximum area.

115. Having given the hypotenuse of a right-angled triangle, describe the triangle of maximum area.

116. Through a given point within a circle draw the minimum and the maximum chords.

117. A circle is drawn through two given points touching a given straight line; find the point in the given straight line at which the line joining the two given points subtends the maximum angle.

118. From a given point without a circle, whose centre is O, draw a straight line to cut the circumference in P and Q so that the triangle OPQ may be a maximum.

119. Prove that the greatest rectangle that can be inscribed in a given circle is a square.

120. If the sides of a polygon are not all equal there may be found another polygon having the same number of sides and the same area, but having a less perimeter.

121.* One circle is wholly within another, and contains the centre of the other; find the least and greatest chords of the outer circle which touch the inner circle.

122.* ABC is a given triangle and P is a point in its base; if two lines be drawn through P parallel to the sides of the triangle, show that the parallelogram so formed is greatest when P is taken at the middle of the base.

123.* Of all triangles with given base and area, the isosceles has the least perimeter, but the greatest vertical angle.

124.* The lines joining the feet of the perpendicular drawn from the angles to the opposite sides of the triangle, form an inscribed triangle whose perimeter is a minimum.

125.* Of all triangles that can be inscribed in a circle, the equilateral has the greatest area.

126.* Of all quadrilateral figures that can be inscribed in a circle, the square has the greatest area.

127.* Of all polygons of a given number of sides that can be inscribed in a circle, the regular polygon has the greatest area.

128.* Of all triangles that can be inscribed in a circle, the equilateral has the greatest perimeter.

MISCELLANEOUS.

129. Draw through P a straight line terminated by two intersecting lines OA, OB and bisected at P, by joining OP and producing to Q so that $PQ = OP$.

130. In any triangle an angle is acute, right, or obtuse, according as the median drawn from that angle is greater than, equal to, or less than half the side opposite the angle.

131. The straight line AB is bisected at C, and AX, BY, CZ are drawn perpendicular to any other straight line PQ, which does not pass between A and B; prove that CZ is equal to half the sum of AX and BY.

What difference would it make if PQ did pass between A and B?

132. ABC is a triangle of which AX is the median, AD is the bisector of the vertical angle, and AP is the perpendicular on BC; if AB is greater than AC, prove that BX is less than BD, and BD less than BP.

133. If two medians are equal, the triangle is isosceles.

134.* If on the same base and on the same side of it there be described any number of triangles with equal vertical angles, prove that the internal bisectors of the vertical angles all pass through one fixed point, and the external bisectors of the vertical angles through another fixed point.

135.* If two triangles be on equal bases and between the same parallels, the intercepts made by the sides of the triangles on any straight line parallel to the base are equal.

136.* Three circles, whose centres are A, B, C, touch one another externally, two at a time, at D, E, F; show that the in-circle of the triangle ABC is the circum-circle of the triangle DEF.

137.* If the diameter of one of the described circles of a triangle is equal to the perimeter of the triangle, show that the triangle is right-angled.

138.* The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

139.* If the arms of the vertical angle of a triangle, whose perimeter is constant, are given in position, then the base always touches a fixed circle.

140.* Having given the hypotenuse of a right-angled triangle and the radius of the inscribed circle, construct the triangle.

141.* Given the base, the vertical angle, and the radius of the escribed circle opposite to the vertex, construct the triangle.

142.* The bisector of the angle A of the triangle ABC meets the internal and external bisectors of the angle C in P and Q, and the circum-circle of the triangle ABC in R; prove that R is equidistant from P and Q.

143.* From a given point draw a line to form with a given angle a triangle of given perimeter.

144.* A triangle of constant size and shape moves so that two of its sides pass through two fixed points. Show that the third side touches a fixed circle.

145.* The lines drawn from the centres of the three escribed circles perpendicular to the sides they touch are concurrent.

CHAPTER XV.

THE COMPOSITION OF RECTANGLES.

227. **Squares and Rectangles.**—This Chapter contains several theorems on the areas of rectangles and squares. These theorems are obtained by dividing rectangles and squares into smaller rectangles and squares, and thus showing how the original figures can be built up from these parts.

If we are given any straight line X, and any number of other straight lines each equal to X, then the squares on these lines are all congruent (Pm.7) and therefore equal in area: thus we may refer to any one of these squares as the "square on X." Similarly if we are given any two straight lines X, Y, and a set of rectangles each of which has two adjacent sides equal to X, Y respectively, then (Pm.7) these rectangles are all congruent and therefore equal in area: thus we may refer to any one of these rectangles as the "rectangle (X, Y)."

These facts are sometimes expressed by saying that the square on X is *unique* (*i. e.* has one definite shape and size), and the rectangle (X, Y) is *unique*.

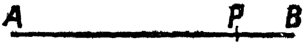

FIG. 347.

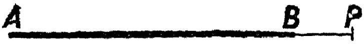

FIG. 348.

228. **Segments.**—In Fig. 347 the point P is said to divide the line AB **internally**: in Fig. 348, the point P (which lies in AB *produced*) is said to divide the line **externally.** In Fig. 347 AB is equal to the **sum** of AP, PB: in Fig. 348 AB is equal to the **difference** of AP, PB.

In Fig. 347 AP, PB are the *parts* of the line AB: in Fig. 348 AP, PB are *not* parts of the line AB. In many theorems, however, it is convenient to use some term which will apply to the lines AP, PB whether P lies in AB or in AB produced, and the word **segment** is used for this purpose. Thus in Fig. 347 we say that P divides AB *internally* into *segments* AP, PB; and in Fig. 348 we say that P divides AB *externally* into *segments* AP, PB.

The word segments may also be applied to areas. Thus in Fig. 349 we may say that the line FG divides the rectangle $ABCD$ *internally* into the segments $ABGF$, $FGCD$; or that the line LM divides the rectangle $ABCD$ *externally* into the segments $ABML$, $LMCD$.

EXERCISES LXXXIV (*Riders*).

1. A straight line is divided internally at a given point: show that the difference between the segments is double of the distance of the given point from the middle of the line.

2. A straight line is divided externally at a given point: show that the sum of the segments is double of the distance of the given point from the middle of the line.

229. THEOREM [R.1].—**If a rectangle be divided (internally or externally) by lines parallel to its sides then any of the quadrilaterals so formed is a rectangle, and its opposite sides are equal.**

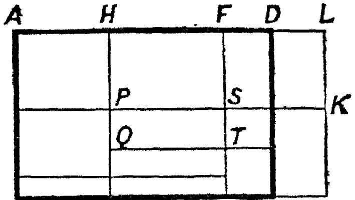

FIG. 349.

In Fig. 349 :—

Given that rect. $ABCD$ is divided (internally or externally) by lines parallel to AB and AD,

Required to prove that any of the resulting quadrilaterals (*e.g.* $PQTS$, or $HPKL$) is a rectangle, and that its opposite sides are equal.

Proof.—The lines drawn $\parallel AD$, are all $\parallel$ one another. The lines drawn $\parallel AB$ are all $\parallel$ one another. P.3.

$\because PS \parallel QT$ and $PQ \parallel ST$, $\therefore PQTS$ is $\parallel^{\text{m}}$ *Def.*

Hence $PS = QT$, $PQ = ST$. Pm.1.

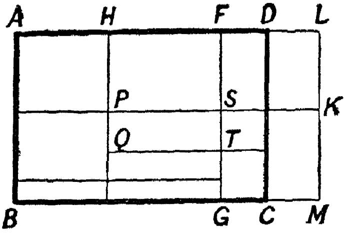

FIG. 349.

Again PQ, PS are respectively parallel to and in the same sense as AB, AD.

$\therefore \angle SPQ = \angle A = \text{R}$. P.4.

Hence $PQTS$ is a rect. Pm.3.

Similarly $HPKL$ is a rect., and $HP = LK$, $HL = PK$.

COROLLARY.—**If a rectangle be divided (internally or externally) by lines perpendicular to its sides then any of the quadrilaterals so formed is a rectangle.**

For lines which are perpendicular to one side of a rectangle are parallel to two other sides. P.2.

EXERCISES LXXXV.

(*In these exercises proofs are not required.*)

1. In Fig. 350 P, Q, X, Y denote the lines AD, DG, AB, BC, respectively, or lines equal to them in length †; thus the rectangle $ACKG$ is rect. $(P+Q, X+Y)$; also rect. $EFKH$ is rect. (Q, Y), for it is contained by two lines EH, EF which are respectively equal to DG, BC. Name the rectangles AE, DH, BF, in a similar manner.

Hence complete the following statement—

rect. $(P+Q, X+Y)$ = rect. (P, X) + . . . + . . . + . . .

---

† Note that P, Q, X, Y denote lines, not *measures* of lines in terms of some unit of length. The distinction is of great importance: for so long as we are dealing with lines and not measures, it does not matter whether these lines are or are not commensurable to each other or to any unit of length.

2. In Fig. 351 P, Q, X, Y, Z denote the segments of the sides of the rectangle. Draw a rough copy of the figure and in each of the small rectangles write its name. Hence express rect. $(P+Q, X+Y+Z)$ as the sum of six rectangles.

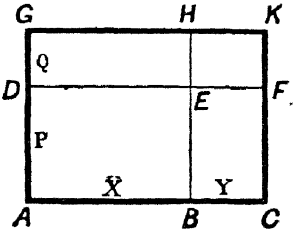

FIG. 350.

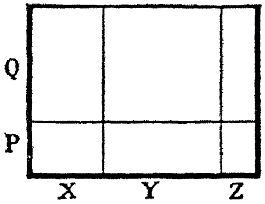

FIG. 351.

3. Write down an equation (involving rectangles and squares) corresponding to each of the diagrams in Fig. 352. The letters P, Q . . . denote the segments of the sides. [If you find any figure difficult, draw a rough copy and in each rectangle write its name.]

4. Using Fig. 355 express rect. $(P, X-Y)$ as the difference of two rectangles.

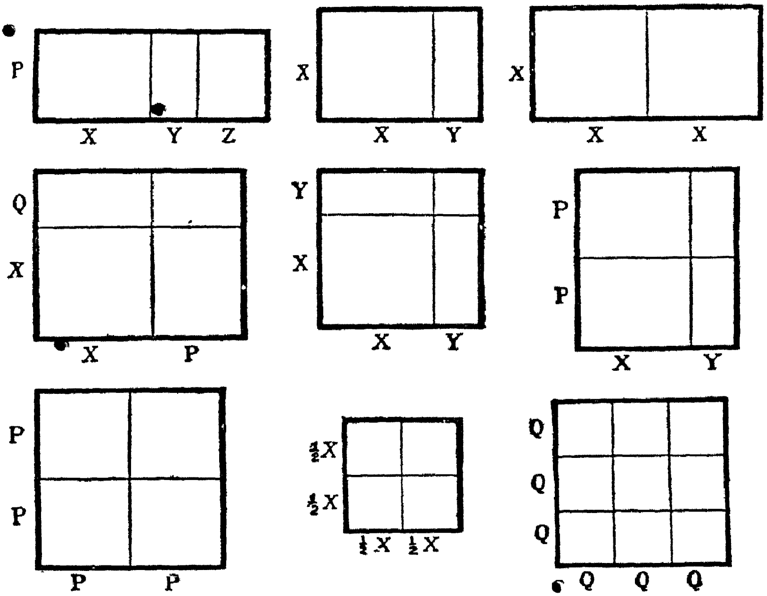

FIG. 352.

230. THEOREM [R.2].—**The rectangle contained by any given line and the sum (or difference) of two other given lines is equal to the sum (or difference) of the rectangles contained by the first line and each of the other two lines.**

In Fig. 353 :—

P, X, Y,† are any three straight lines, and $X > Y$:—

Required to prove that

(i) rect. $(P, X+Y)$ = rect. (P, X) + rect. (P, Y),

(ii) rect. $(P, X-Y)$ = rect. (P, X) − rect. (P, Y).

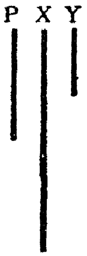

FIG. 353.

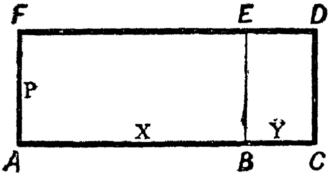

FIG. 354.

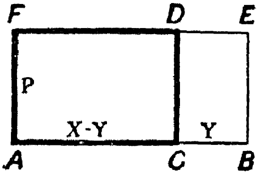

FIG. 355.

Cons.—Draw $AB = X$, and take a point C in AB produced (Fig. 354), or in AB (Fig. 355), so that $BC = Y$.

Draw $AF \perp AC$ and equal to P.

Complete the rect. $AFDC$, and draw $BE \parallel AF$.

Proof.—In either figure :—

$\because$ $BE \parallel AF$ which is a side of rect. AD,

$\therefore$ quads. AE, CE are rects. and $BE = AF = P$. R.1.

Hence AE = rect. (P, X), CE = rect. (P, Y).

(i) In Fig. 354 :—

$\because$ $AC = X+Y$, $\therefore$ rect. AD = rect. $(P, X+Y)$.

Also rect. AD = rect. AE + rect. CE;

hence rect. $(P, X+Y)$ = rect. (P, X) + rect. (P, Y).

(ii) In Fig. 355 :—

$\because$ $AC = X-Y$, $\therefore$ rect. AD = rect. $(P, X-Y)$.

Also rect. AD = rect. AE − rect. CE;

hence rect. $(P, X-Y)$ = rect. (P, X) − rect. (P, Y).

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† See footnote on page 332.

COROLLARY.—**The rectangle contained by any given line and the sum of any number of other given lines is equal to the sum of the rectangles contained by the first line and each of the other lines.**

In Fig. 356, rect. AK = rect. $AG + BH + CK$,
hence rect. $(P, X+Y+Z)$
= rect. (P, X) + rect. (P, Y) + rect. (P, Z).

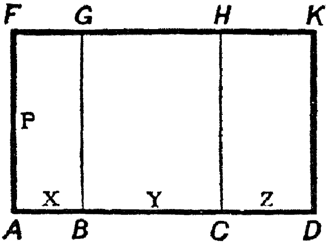

FIG. 356.

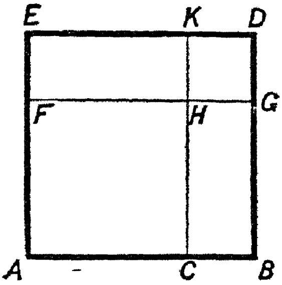

FIG. 357.

231. THEOREM [R.3].—**If a line be divided *internally* at any point, then the square on the line is equal to the sum of the squares on the segments *plus* twice the rectangle contained by the segments.**

In Fig. 357, AB is divided internally at C :—
Required to prove that
sq. on AB = sq. on AC + sq. on CB + 2 rect. (AC, CB).

Cons.—On AB draw sq. $ABDE$.
From AE mark off $AF = AC$; then $FE = CB$.
Draw $CK \parallel AE$ and $FHG \parallel AB$.

Proof.—∵ sq. AD has been divided by lines drawn parallel to its sides,

∴ every quad. in the figure is a rect. (or sq.), and opposite sides of any quadrilateral are equal. R. 1.

∵ $AF = AC$, ∴ rect. AH = sq. on AC.
∵ $HG = CB$, and $HK = FE = CB$, ∴ rect. HD = sq. on CB.
∵ $CH = AF = AC$, ∴ rect. CG = rect. (AC, CB).
∵ $FH = AC$ and $FE = CB$, ∴ rect. FK = rect. (AC, CB).

Now sq. AD = sq. AH + sq. HD + rect. CG + rect. FK ;
thus sq. on AB = sq. on AC + sq. on CB + 2 rect. (AC, CB).

COROLLARY 1.—**The square on the sum of two lines is equal to the sum of the squares on the two lines *plus* twice the rectangle contained by the two lines.**

For in Fig. 357 AB is the sum of the lines AC, CB.

COROLLARY 2.—**The square on any line is equal to four times the square on half the line.**

For in Fig. 357, if C bisects AB, then each rectangle will be equal to the sq. on AC.

232. THEOREM [R.4].—**If a line be divided *externally* at any point, then the square on the line is equal to the sum of the squares on the segments *minus* twice the rectangle contained by the segments.**

In Fig. 358 AB is divided externally at C :—

Required to prove that

sq. on AB = sq. on AC + sq. on BC − 2 rect. (AC, BC).

Cons.—On AC draw sq. $ACDE$.

From AE mark off $AF = AB$; then $FE = BC$.

Draw $BK \parallel AE$ and $FGH \parallel AC$.

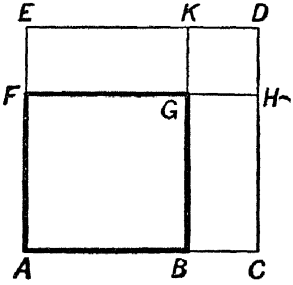

FIG. 358.

Proof.—∵ sq. AD has been divided by lines drawn parallel to its sides, ∴ every quad. in the figure† is a rect. (or sq.), and has its opposite sides equal. R.1

∵ $AF = AB$, ∴ rect. AG = sq. on AB.

∵ $GH = BC$, and $GK = FE = BC$, ∴ rect. GD = sq. on BC.

∵ $CD = AC$, ∴ rect. BD = rect. (AC, BC).

∵ $FH = AC$ and $FE = BC$, ∴ rect. FD = rect. (AC, BC).

Hence sq. on AB + 2 rect. (AC, BC)

= sq. AG + rect. BD + rect. FD

= sq. AG + rect. BD + rect. FK + sq. GD

= sq. AD + sq. GD

= sq. on AC + sq. on BC.

Thus sq. on AB = sq. on AC + sq. on BC − 2 rect. (AC, BC).

---

† See footnote on p. 337.

COROLLARY.—**The square on the difference of two lines is equal to the sum of the squares on the two lines *minus* twice the rectangle contained by the two lines.**

For, in Fig. 358, AB is the difference between AC, CB.

EXERCISES LXXXVI (Riders).

(*In Questions* 1–4 *prove direct from a figure.*)

1. Rect. $(P+Q, X+Y)$
$=$rect. $(P, X)+$rect. $(Q, X)+$rect. $(P, Y)+$ rect. (Q, Y).
2. Sq. on $(P+2Q)=$sq. on $P+4$ sq. on $Q+4$ rect. (P, Q).
3. Rect. $(P+Q, X-Y)$
$=$rect. $(P, X)+$rect. $(Q, X)-$rect. $(P, Y)-$rect. (Q, Y).
4. Rect. $(P-Q, X-Y)$
$=$rect. $(P, X)+$rect. $(Q, Y)-$rect. $(P, Y)-$rect. (Q, X).

(In Questions 5–10 give only *the figure and the results* without proofs.)

5. Rect. $(P, 2Q)$. 6. Rect. $(2P, 3Q)$. 7. Sq. on $4P$.
8. Rect $(P+Q, P+2Q)$. 9. Rect. $(P+2Q, 2P+Q)$.
10. Rect. $(P+Q, P+Q+R)$.

233. *Prel. Ex.*—In Fig. 359 name the rectangles FD, CG in terms of the lengths X, Y. These same two rectangles can be joined so as to form one rectangle; draw a figure to represent this, and name this one rectangle in terms of the lengths X, Y.

Appeal to the figure.—It is worthy of notice that in theorems R.1—R.5 we assume *on the evidence of the figures* that the constructions used divide the given rectangles into *quadrilaterals* (placed in certain relative positions with respect to one another) and do *not* divide them into triangles, pentagons, etc.

This **appeal to the figure** is no new element in the line of geometrical argument, and many instances can be quoted from earlier chapters. Thus in Theorem A.2 we have assumed *from the figure* that two intersecting lines form four angles: the proof of Theorem P.4 rests on the fact that no other type of figure *can be drawn* than those given in the text: the proof of Theorem T.1 assumes that the line CE does not fall *without* the angle ACD: and so on. In short, it is usual to accept the drawings themselves as sufficient demonstration of the *type* or *types* of figure which can be produced by the given constructions.

This practice may of course be regarded as a serious lapse in logic, but it can only be avoided by adopting several *new axioms* and by greatly complicating all geometrical argument. The axioms required are given in Prof. Hilbert's *Foundations of Geomtery* (Chapter I.)—a book which forms a splendid exercise in pure logic. But it is only necessary to read these axioms and the illustrations of their use given by the author to be convinced that the appeal to the figure is a necessity in school work and highly desirable in all geometry.

THEOREM [R.5].—**The difference between the squares on two given lines is equal to the rectangle contained by the sum and the difference of the two given lines.**

In Fig. 359 X, Y are two lines of which $X > Y$.

Required to prove that

sq. on $X -$ sq. on $Y =$ rect. $(X+Y, X-Y)$.

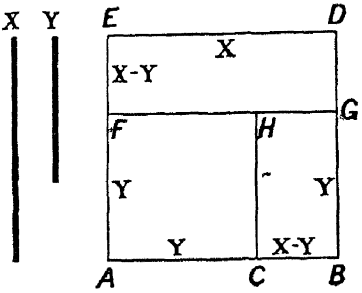

FIG. 359.

Cons.—Draw $AB = X$, and from AB mark off $AC = Y$; then $BC = X - Y$. On AB draw the square $ABDE$.

From AE mark off $AF = Y$; then $FE = X - Y$.

Draw $FG \parallel AB$, and $CH \parallel AE$.

Proof.—$\because$ AD has been divided by lines drawn parallel to its sides, $\therefore$ every quad. in the figure is a rect. (or sq.), and opposite sides of any quad. are equal. R.1.

$\because$ $AF = AC = Y$, $\therefore$ rect. $AH =$ sq. on Y.

$\because$ $ED = AB = X$, $\therefore$ rect. $FD =$ rect. $(X - Y, X)$;

$\because$ $CH = AF = Y$, $\therefore$ rect. $CG =$ rect. $(X - Y, Y)$.

Thus sq. on $X -$ sq. on $Y =$ sq. $AD -$ sq. AH,

$=$ rect. $FD +$ rect. CG,

$=$ rect. $(X-Y, X) +$ rect. $(X-Y, Y)$.

But, by Theorem R.2,

rect. $(X-Y, X) +$ rect. $(X-Y, Y) =$ rect. $(X-Y, X+Y)$.

Hence sq. on $X -$ sq. on $Y =$ rect. $(X+Y, X-Y)$.

COROLLARY 1.—**If a line be divided equally and also unequally, then the rectangle contained by the unequal segments is equal to the square on half the line minus the square on the line between the points of section.**

In Fig. 360, C bisects AB, and D divides AB unequally. Then $AD = AC + CD$, and $DB = CB - CD = AC - CD$,
Hence rect. $(AD, DB) =$ rect. $(AC + CD, AC - CD)$
$=$ sq. on $AC -$ sq. on CD. R.5.

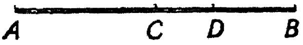

FIG. 360.

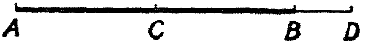

FIG. 361.

COROLLARY 2.—**If a line is bisected and produced to any point, then the rectangle contained by the whole line and the part produced is equal to the square on the line made up of the half and the part produced minus the square on half the line.**

In Fig. 361, AB is bisected at C and produced to D. Then $AD = CD + AC$, and $BD = CD - CB = CD - AC$,
Hence rect. $(AD, BD) =$ rect. $(CD + AC, CD - AC)$
$=$ sq. on $CD -$ sq. on AC. R.5.

234. THEOREM [R.6].—**The sum of the squares on the sum and difference of two lines is equal to twice the sum of the squares on the two lines; also the difference of the squares on the sum and difference of two lines is equal to four times the rectangle contained by the two lines.**

In Fig. 362, X, Y are any two lines of which $X > Y$:—

FIG. 362.

Required to prove that

(i) sq. on $(X+Y) +$ sq. on $(X-Y)$
$= 2$ sq. on $X + 2$ sq. on Y;

(ii) sq. on $(X+Y) -$ sq. on $(X-Y) = 4$ rect. (X, Y).

Proof.—By Theorems R.3 and R.4 we have

sq. on $(X+Y) =$ sq. on $X +$ sq. on $Y + 2$ rect. (X, Y) (a)
sq. on $(X-Y) =$ sq. on $X +$ sq. on $Y - 2$ rect. (X, Y) (b)

In equations (a) and (b) add equals to equals; **thus** sq. on $(X+Y) +$ sq. on $(X-Y) = 2$ sq. on $X + 2$ sq. on Y.
In equations (a) and (b) subtract equals from equals, **thus** sq. on $(X+Y) -$ sq. on $(X-Y) = 4$ rect. (X, Y).

EXERCISES LXXXVII (Riders).

1. Prove R.3, Cor. 1 from R.2.
2. Prove R.4, Cor. from R.2.
3. Prove R.5 from R.2.
4. Using R.2 express sq. on $P+Q+R$ as the sum of certain squares and rectangles.

235. **Equivalent Algebraical Formulæ.**—The Theorems which we have just proved (R.2—R.6) express the area of various rectangles or squares in terms of the areas of other rectangles or squares. Now we know (from Ar.1) that the area of a rectangle is the product of the lengths of two adjacent sides, and the area of a square is the square (in the algebraical sense) of the length of its side. Hence, using these theorems, we can express various algebraical squares and products in terms of other algebraical squares and products, *i.e.* we can deduce certain algebraical results in multiplication. This algebraical interpretation of theorems R.2—R.6 is of considerable importance, and is worked out in detail in Exercises LXXXVIII.

Of course this method does not prove these formulæ for incommensurable numbers till Case III. of Theorem Ar.1 has been established.

The reverse process is also possible, viz. to deduce these geometrical theorems from the corresponding algebraical formulæ. For this reason some teachers prefer to treat Theorems R.2—R.6 as algebraical theorems which are capable of geometrical illustration. For the sake of such we repeat the theorems in this form in § 236–239 and Exercises LXXXIX.

It is, however, to be noted that these algebraical theorems are in all elementary books proved true only for commensurable magnitudes and their truth for incommensurable magnitudes is assumed. Thus we do not get a logically valid treatment of these theorems by appealing to Algebra—at all events to the Algebra which has been mastered by the student at this stage. The authors of this book therefore regard the geometrical treatment of theorems R.2—R.6 as preferable to the algebraical, for the former is logically complete.

EXERCISES LXXXVIII.

1. In Fig. 363 suppose that the lines *AF, AB, BC, CD* contain respectively p, x, y, z units of length; how many units of area are there in the rectangles *AG, BH, CK, AK*?

Now use Theorem R.2, Cor. to establish the algebraical formula

$$p(x+y+z)=px+py+pz$$

What algebraical formulæ can be deduced from the following theorems?—

2. R.3. (Let the two *segments* contain x and y units of length).

3. R.4 (see hint for Ques. 2). **4.** R.5. **5.** R.6.

ALTERNATIVE TREATMENT OF THEOREMS R.2—R.6.

236.† Geometrical illustration of the formula

$$p\,(x+y+z)=px+py+pz.$$

[This formula corresponds to Theorem R.2. Cor.]

Cons.—In Fig. 363 let *ADKF* be a rectangle divided by lines *BG, CH*, each ∥ *AF*.

Let *AF, AB, BC, CD* contain p, x, y, z units of length respectively.

Proof.—∵ *BG, CH* are ∥ *AF*,

∴ each of the quads. *AG, BH, CK* is a rect., and *BG* = *CH* = *AF*. R.1.

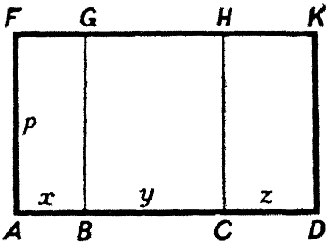

FIG. 363.

∵ *AF* contains p units of length, and *AD* contains $(x+y+z)$ units of length,

∴ rect. *AK* contains $p\,(x+y+z)$ units of area. Ar.1.

Similarly the rects. *AG, BH, CK* contain respectively px, py, pz units of area.

But rect. *AK* = rect. *AG* + rect. *BH* + rect. *CK*.

Hence $p(x+y+z)= px + py + pz$

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† §§ 236–239 should be either omitted or used *instead of* §§ 230–235 and Exercises LXXXVIII. Exercises LXXXIX should however be worked by *all* students.

237. Geometrical illustration of the formula

$$(x+y)^2 = x^2+y^2+2xy.$$

[This formula corresponds to Theorem R.3.]

Cons.—In Fig. 364 let $ABDE$ be a square, divided by lines CK, FG parallel to the sides.

Also let $AF=AC$: then $FE=CB$.

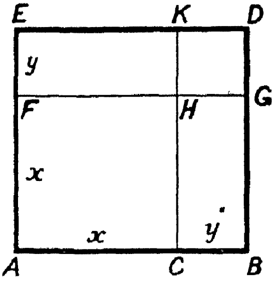

FIG. 364.

Let AC, CB contain x, y units of length respectively. Then AB contains $(x+y)$ units of length.

Proof.—$\because$ FG, CK are ∥ the sides of the square,

$\therefore$ each quad. in the figure is a rect. and has its opposite sides equal. R.1.

$\because AF=AC=x$,

$\therefore$ sq. $AH=x^2$ units of area; Ar.1.

$\because HG=CB=y$, and $HK=FE=CB=y$,

$\therefore$ sq. $HD=y^2$ units of area; Ar.1.

$\because FH=AC=x$, and $FE=CB=y$,

$\therefore$ rect. $FK=xy$ units of area; Ar.1.

$\because CH=AF=AC=x$, and $CB=y$,

$\therefore$ rect. $CG=xy$ units of area; Ar.1.

$\because AE=AB=(x+y)$,

$\therefore$ sq. $AD=(x+y)^2$ units of area. Ar.1.

But sq. $AD=$ sq. $AH+$ sq. $HD+$ rect. $FK+$ rect. CG

hence $(x+y)^2 = x^2 + y^2 + 2xy$.

238. Geometrical illustration of the formula $(x-y)^2=x^2+y^2-2xy$.

[This formula corresponds to Theorem R.4.]

Cons.—In Fig. 365 let $ACDE$ be a square, divided by lines BK, FH parallel to the sides.

Also let $AF=AB$; then $FE=BC$.

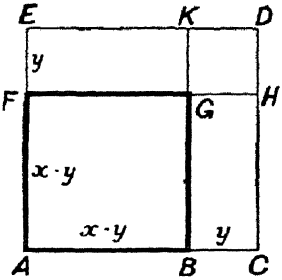

FIG. 365.

Let AC, BC contain x, y units of length respectively. Then AB contains $(x-y)$ units of length.

Proof.—$\because$ FH, BK are $\parallel$ sides of the square,
$\therefore$ each quad. in the figure is a rect. and has its opposite sides equal. R.1.

$\because$ $AE=AC=x$, $\therefore$ sq. $AD=x^2$ units of area ; Ar.1.
$\because$ $GH=BC=y$, and $GK=FE=BC=y$,
$\therefore$ sq. $GD=y^2$ units of area ; Ar.1.
$\because$ $FH=AC=x$, and $FE=BC=y$,
$\therefore$ rect. $FD=xy$ units of area ; Ar.1.
$\because$ $BK=AE=AC=x$, and $BC=y$,
$\therefore$ rect. $BD=xy$ units of area ; Ar.1.
$\because$ $AF=AB=(x-y)$,
$\therefore$ sq. $AG=(x-y)^2$ units of area. Ar.1.

Now sq. AG + rect. FD + rect. BD
= sq. AG + rect. FD + rect. BH + sq. GD
= sq. AD + sq. GD

thus $(x-y)^2+xy+xy = x^2+y^2$,

that is $(x-y)^2 = x^2+y^2-2xy$.

239. Geometrical illustration of the formula

$$x^2 - y^2 = (x+y)(x-y).$$

[This formula corresponds to Theorem R.5.]

Cons.—In Fig. 366, let $ABDE$ be a square divided internally and externally by lines CH, FL, KL, which are ∥ sides of the square.

Also let $DK = AF = AC$; then $FE = CB$.

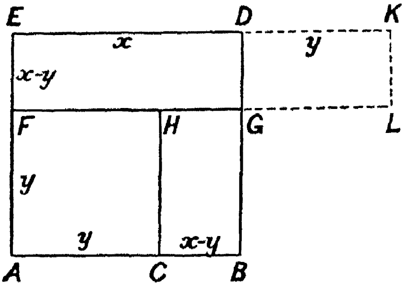

FIG. 366.

Let AB, AC contain x, y units of length respectively; then CB contains $(x-y)$ units of length.

Proof.—$\because$ CH, FL, KL are ∥ sides of sq. AD,

$\therefore$ every quad. in the figure is a rect. and has its opposite sides equal. R.1.

$\because$ $DG = FE = CB$, and $DK = AF = CH$,

$\therefore$ rect. $GK \equiv$ rect. CG. Pm.7.

$\because$ $ED = AB = x$, $DK = AC = y$, $\therefore$ $EK = (x+y)$;

also $FE = CB = (x-y)$;

$\therefore$ rect. $FK = (x+y)(x-y)$ units of area. Ar.1.

$\because$ $AE = AB = x$, $\therefore$ sq. $AD = x^2$ units of area; Ar.1.

$\because$ $AF = AC = y$, $\therefore$ sq. $AH = y^2$ units of area. Ar.1.

Now sq. AD − sq. AH = rect. FD + rect. CG

= rect. FD + rect. GK,

that is sq. AD − sq. AH = rect. FK;

Hence $x^2 - y^2 = (x+y)(x-y)$.

EXERCISES LXXXIX.

Illustrate the following algebraical formulæ by geometrical figures :—

1. $p(x+y) = px + py$. 2. $p(x-y) = px - py$.

3. $(p+q)(x+y) = px + py + qx + qy$.

4. $(p+q)(x-y) = px + qx - py - qy$.

5. $(p-q)(x-y) = px + qy - py - qx$.

6. $(p+q+r)(x+y) = px + qx + rx + py + qy + ry$.

7. $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.

8. What algebraical formula is illustrated by Fig. 367.

9. Use Fig. 368 to illustrate the algebraical formula

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2.$$

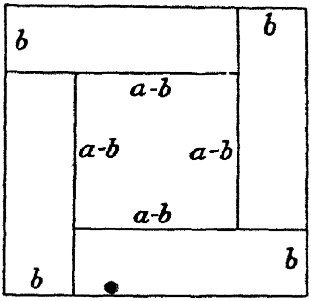

FIG. 367.

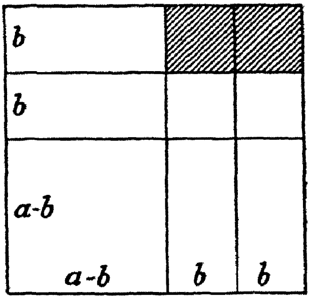

FIG. 368.

RIDERS.

10. [R.7] **If a line is divided equally and unequally, then the sum of the squares on the unequal segments is equal to twice the sum of the squares on the half-line and the line between the points of section.**

11. [R.8] **If a line is bisected and produced then the sum of the squares on the whole line and on the part produced is equal to twice the sum of the squares on half the original line and on the line made up of the half and the part produced.**

12. If A, B, C, D be four points taken in order on a straight line prove that rect. (AB, CD) + rect. (BC, AD) = rect. (AC, BD).

13. [RT.3] **In a right-angled triangle the square on the line drawn perpendicular to the hypotenuse from the opposite vertex is equal to the rectangle contained by the segments into which this line divides the hypotenuse.** [R.3.

14.* Squares are described on two adjacent sides of a rectangle : show that the rectangle contained by the diagonals of these squares is double the area of the original rectangle. [A.1c ; RT.2 ; R.3.

CHAPTER XVI.

RECTANGLE-THEOREMS ON TRIANGLES AND CIRCLES.

240. **Notation.**—The area of the rectangle contained by any two lines is the product of the lengths of these lines (Ar.1), and the area of the square on any line is the (algebraical) square of the length of this line. As then algebraic squares and products correspond to geometrical squares and rectangles it is convenient to use the algebraic notations to denote the corresponding geometrical figures. Thus throughout the remainder of this book we shall use the symbol AB^2 to denote the square on the line AB, and the symbol $AB.CD$ to denote the rect. contained by the lines AB and CD. No confusion of ideas need arise from this practice: for example, if X and Y represent two lines, and x and y represent two numbers, then X, Y will represent a *rectangle*, but xy will represent a *product*.

Revise carefully Theorem RT.2, which proves the relation between the areas of the squares on the sides of a right-angled triangle.

Theorems T.8 and T.9 establish the corresponding results for acute-angled and obtuse-angled triangles.

EXERCISES XC.

1. Draw a triangle ABC, making (say) $AB=1{\cdot}8''$, $BC=2''$, $CA=1{\cdot}5''$. On AB, BC, CA construct the squares $ABNP$, $BCHK$, $CALM$. (*Cf.* Fig. 369.) Draw $ADX \perp BC$, meeting BC and KH at D and X: draw $BEY \perp CA$, meeting CA and ML at E and Y: draw $CFZ \perp AB$, meeting AB and PN at F and Z.

Find the area of each of the rectangles AZ, BZ, BX, CX, CY, AY.

What do you notice about the results?

2. Each of the following statements applies to Fig. 369: can you fill in the blanks?

(i) $AB^2 = CA^2 + CB^2 - (CA. \quad\;) - (CB. \quad\;)$.
(ii) $AB^2 = CA^2 + CB^2 - 2\,(CA. \quad\;)$.
(iii) $AB^2 = CA^2 + CB^2 - 2\,(CB. \quad\;)$.
Use the results of Ques. 1.

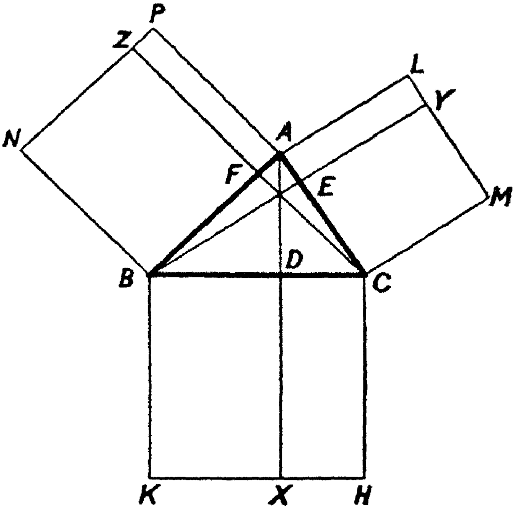

FIG. 369.

3. Draw a triangle ABC making (say) $AB = 5$ cms., $BC = 4$ cms., $CA = 2$ cms. Using this triangle repeat Ques. 1, following the same instructions in every particular, and *producing* the lines where necessary. Note carefully the modifications in the figure caused by the change in the shape of the triangle.

Find the area of each of the rectangles AZ, BZ, BX, CX, CY, AY.
What do you notice about the results?

4. Each of the following statements applies to the figure in Ques. 3. Using your results in Ques. 3, can you fill in the blanks?

(i) $AB^2 = CA^2 + CB^2 + (CA. \quad\;) + (CB. \quad\;)$.
(ii) $AB^2 = CA^2 + CB^2 + 2\,(CA. \quad\;)$.
(iii) $AB^2 = CA^2 + CB^2 + 2\,(CB. \quad\;)$.

5*. Each of the following statements applies to the figure in Ques. 3. Using your results in Ques. 3, can you fill in the blanks?

(i) $BC^2 = AB^2 + AC^2 - (AB. \quad\;) - (AC. \quad\;)$.
(ii) $BC^2 = AB^2 + AC^2 - 2\,(AB. \quad\;)$.
(iii) $BC^2 = AB^2 + AC^2 - 2\,(AC. \quad\;)$.

241. THEOREM [T.8].—**In an *obtuse-angled* triangle the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle *plus* twice the rectangle contained by one of these two sides and the projection of the other upon it.**

In $\triangle ABC$ (Fig. 370) :—
Given that $\angle C$ is obtuse, and that $AD \perp BC$,

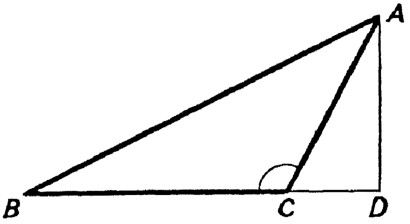

FIG. 370.

Required to prove that

$$AB^2 = CA^2 + CB^2 + 2\,CB.CD.$$ [†]

Proof.—$\because$ line BD is divided internally at C,[‡]

$$\therefore BD^2 = CB^2 + CD^2 + 2\,CB.CD. \qquad \text{R.3.}$$

To each of these equals add AD^2;

$$\text{thus} \quad BD^2 + AD^2 = CB^2 + CD^2 + AD^2 + 2\,CB.CD. \ldots\ldots \quad (a)$$

$$\text{Since } \angle D = \text{R}, \therefore BD^2 + AD^2 = AB^2, \qquad \text{RT.2.}$$

$$\text{and } CD^2 + AD^2 = CA^2. \qquad \text{RT.2.}$$

Apply these two results in equation (a);

$$\text{thus} \qquad AB^2 = CB^2 + CA^2 + 2\,CB.CD.$$

[†] In Theorems T.8 and T.9 the name of each line mentioned on the right side of the equation commences with the letter C. This is a useful mnemonic.

[‡] The student should notice the tacit assumption in Theorem T.8 (Fig. 370) that if $\angle ACB$ is obtuse, then D falls in *BC produced*; also the corresponding assumption in Theorem T.9 (Figs. 371, 372) that if angles B and C are both acute then D falls in BC, but that if $\angle ABC$ is obtuse then D falls in *CB produced*. These facts can be proved without serious difficulty, but this is generally reckoned a legitimate case for an "appeal to the figure." (See footnote on p. 337.)

242. THEOREM [T.9].—**In *any* triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the two sides containing that angle *minus* twice the rectangle contained by one of these two sides and the projection of the other upon it.**

In $\triangle ABC$ (Figs. 371, 372, 373):—
Given that $\angle C$ is acute and that $AD \perp BC$,
Required to prove that

$$AB^2 = CA^2 + CB^2 - 2\,CB.CD.$$

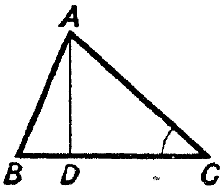

FIG. 371.

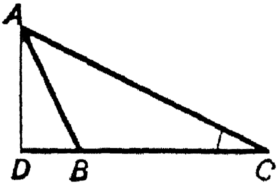

FIG. 372.

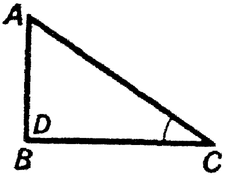

FIG. 373.

Proof.—There are three cases:—
CASE I. Where D falls within BC (Fig. 371).
CASE II. Where D falls in CB produced (Fig. 372).
CASE III. Where D falls at B (Fig. 373).

CASES I and II (Figs. 371, 372).
In each case, ∵ line BD is divided externally at C.

$$\therefore BD^2 = CB^2 + CD^2 - 2\,CB.CD. \qquad \text{R.4.}$$

To each of these equals add AD^2,

thus $$BD^2 + AD^2 = CB^2 + CD^2 + AD^2 - 2\,CB.CD. \ldots\ldots (a)$$

Since $AD \perp BC$, $\therefore\ BD^2 + AD^2 = AB^2$, RT.2.
and $CD^2 + AD^2 = CA^2$. RT.2.

Apply these two results in equation (a);

thus $$AB^2 = CB^2 + CA^2 - 2\,CB.CD.$$

CASE III (Fig. 373).
In this case $\angle ABC = R$.

$$\therefore AB^2 = CA^2 - CB^2 \qquad \text{RT.2.}$$
$$= CA^2 + CB^2 - 2\,CB^2.$$

That is, $$AB^2 = CA^2 + CB^2 - 2\,CB.CD.$$

243. Algebraical proofs.—Using algebra, and Figs. 374 and 375, the preceding proofs of Theorems T.8, T.9 can be considerably shortened. In the following equations, where there are double signs, use the upper signs for Fig. 374 and the lower for Fig. 375:—

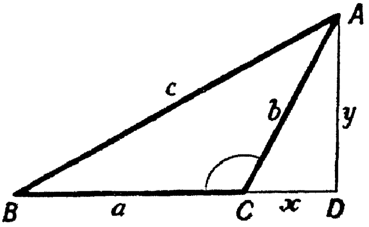

FIG. 374.

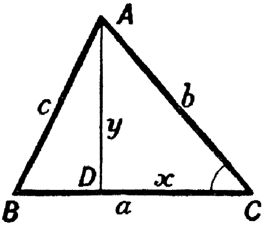

FIG. 375.

By RT. 2, $AD^2 = b^2 - x^2 = c^2 - (a \pm x)^2$.
Thus $b^2 - x^2 = c^2 - a^2 - x^2 \mp 2\,ax$.
Whence $c^2 = a^2 + b^2 \pm 2\,ax$.

244.* Alternative proofs of Theorems T.8, T.9.—The following proofs are rather harder, but much more instructive than the proofs given in §§ 241, 242.

Prel. Ex. In Fig. 376 squares are drawn on the three sides of the triangle ABC: AD, BE, CF, are perpendicular to BC, CA, AB respectively. Prove that (i) $\triangle NBC \equiv \triangle ABK$, and (ii) rect. BZ = rect. BX.

THEOREM [T.8].—**In an** *obtuse-angled* **triangle the square on the side opposite to the obtuse angle is equal to the sum of the squares on the two sides containing the obtuse angle** *plus* **twice the rectangle contained by either of these two sides and the projection of the other upon it.**

In $\triangle ABC$ (Fig. 376):—
Given that $\angle ACB$ is obtuse, and that $AD \perp BC$,
Required to prove that $AB^2 = CA^2 + CB^2 + 2\,CB.CD$.

Cons.—Let AN, BH, CL be the squares on the three sides of the triangle, and let BE be $\perp CA$, and $CF \perp AB$.
Let AD, BE, CF meet HK, LM, NP in X, Y, Z, respectively.
Join AK, CN.†

Proof. (i) $\because$ BH is a square and $DX \perp$ side BC,
$\therefore$ BX and CX are rects. R.1.
Similarly each of the quads. AY, CY, BZ, AZ is a rect. R.1.

---

† The joins of these points are not shown in Fig. 376.

(ii) $\angle NBA = \angle CBK$: for each = R. *Cons.*
To each add $\angle ABC$; thus $\angle NBC = \angle ABK$.

In $\triangle$s NBC, ABK,

$$\because \left\{ \begin{array}{c} NB = AB, \\ BC = BK, \\ \angle NBC = \angle ABK, \end{array} \right\} \therefore \triangle NBC \equiv \triangle ABK. \qquad \text{C.1}$$

$\because$ rect. BZ and $\triangle NBC$ are on the same base NB and between the same parallels NB and ZC,

$\therefore$ rect. $BZ = 2 \triangle NBC$. Ar.3.

Similarly rect. $BX = 2 \triangle ABK$. Ar.3.

But $\triangle NBC \equiv \triangle ABK$;

hence rect. $BZ =$ rect. BX .... (*a*)

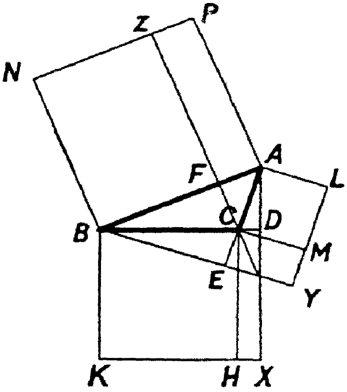

FIG. 376.

Similarly † rect. AZ = rect. AY .... (*b*)

and rect. CX = rect. CY .... (*c*)

(iii) Thus sq. BP = rect. AZ + rect. BZ

= rect. AY + rect. BX [Using (*a*) and (*b*)

= sq. CL + rect. CY + sq. CK + rect. CX

= sq. CL + sq. CK + 2 rect. CX [Using (*c*)

That is $AB^2 = CA^2 + CB^2 + 2\, CB.CD$.

---

† The three results (*a*), (*b*) & (*c*) are most easily remembered by noting (1) that they deal with *rectangles*, not *squares*, and (2) that each vertex of the given triangle is a common vertex of two equal rectangles—*e. g.* A is a common vertex of the equal rectangles AY and AZ.

245.* THEOREM [T.9].—**In *any* triangle the square on a side opposite to an acute angle is equal to the sum of the squares on the two sides containing the acute angle *minus* twice the rectangle contained by either of these two sides and the projection of the other upon it.**

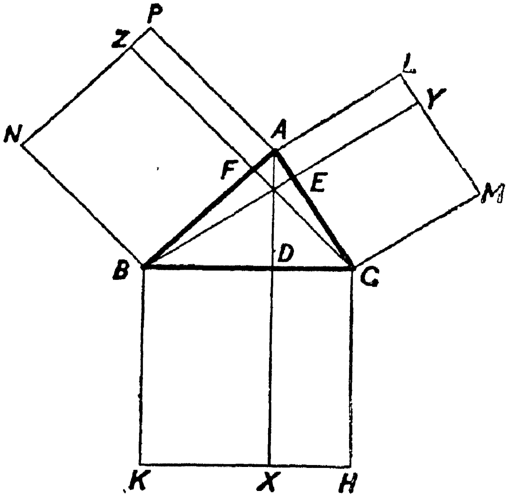

FIG. 377.

In △ *ABC* (Figs. 377, 378) :—

Given that side *AB* is opposite to the acute ∠ *C*, and that $AD \perp BC$,

Required to prove that

$$AB^2 = CA^2 + CB^2 - 2\,CB.CD.$$

Cons.—Let *BH, CL, AN* be the squares on the three sides, and let CF be $\perp AB$, and $BE \perp CA$.

Let *AD, BE, CF* meet *HK, LM, NP* in *X, Y, Z,* respectively. Join *AK, CN*.

Proof.—(i) It can be proved, as in T.8, that each of the quads, *AZ, BZ, AY, CY, BX, CX* is a rect.

(ii) It can be proved, as in T.8, that

$$\triangle NBC \equiv \triangle ABK,$$

and hence that rect. BZ = rect. BX............ (a)
Similarly rect. AY = rect. AZ............ (b)
and rect. CY = rect. CX............ (c)

(iii) CASE I.—When $\triangle ABC$ is acute-angled (Fig. 377).

$$\begin{aligned} \text{sq. } AN &= \text{rect. } AZ + \text{rect. } BZ \\ &= \text{rect. } AY + \text{rect. } BX \quad \text{[Using (a) \& (b)} \\ &= \text{sq. } CL - \text{rect. } CY + \text{sq. } CK - \text{rect. } CX \\ &= \text{sq. } CL + \text{sq. } CK - 2 \text{ rect. } CX \quad \text{[Using (c)} \end{aligned}$$

That is $AB^2 = CA^2 + CB^2 - 2\, CB.CD.$

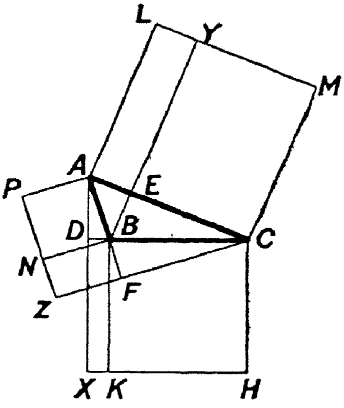

FIG. 378.

CASE II.—When $\triangle ABC$ is obtuse-angled (Fig. 378):—

$$\begin{aligned} \text{sq. } AN &= \text{rect. } AZ - \text{rect. } BZ. \\ &= \text{rect. } AY - \text{rect. } BX \quad \text{[Using (a) \& (b).} \\ &= (\text{sq. } CL - \text{rect. } CY) - (\text{rect. } CX - \text{sq. } CK) \\ &= \text{sq. } CL + \text{sq. } CK - \text{rect. } CY - \text{rect. } CX \\ &= \text{sq. } CL + \text{sq. } CK - 2 \text{ rect. } CX. \quad \text{[Using (c).} \end{aligned}$$

That is $AB^2 = CA^2 + CB^2 - 2\, CB.CD.$

NOTE.—The third case, where $\angle B = \text{R}$, is left as an exercise to the student.

246. THEOREM [T.10].—**The sum of the squares on the two sides of a triangle is equal to twice the sum of the squares on the half of the base and the median which bisects the base. Also, the difference between the squares on the sides of any triangle is equal to twice the rectangle contained by the base and the projection on the base of the median which bisects it.**

In $\triangle ABC$ (Figs. 379, 380) :—
Given that G bisects BC,
Required to prove that

$$\text{(i)}\ AB^2 + AC^2 = 2\,GB^2 + 2\,GA^2,$$
$$\text{(ii)}\ AC^2 - AB^2 = 2\,BC.GD.$$

Cons.—Draw $AD \perp BC$.

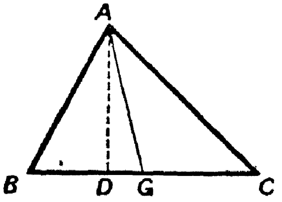

FIG. 379.

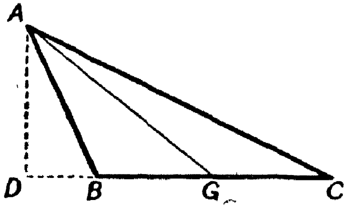

FIG. 380.

Proof.—In $\triangle ACG$, $\because \angle AGC$ is obtuse,
$$\therefore AC^2 = GC^2 + GA^2 + 2\,GC.\,GD. \qquad \text{T.8.}$$
But $GC = GB$, $\therefore GC^2 = GB^2$; Pm.7.
hence
$$AC^2 = GB^2 + GA^2 + 2\,GB.GD. \ldots\ldots\ldots\ldots \quad (a)$$
In $\triangle ABG$, $\because \angle AGB$ is acute, we have (by T.9)
$$AB^2 = GB^2 + GA^2 - 2\,GB.\,GD. \ldots\ldots\ldots\ldots \quad (b)$$
Add equations (*a*) and (*b*)
thus
$$AB^2 + AC^2 = 2\,GB^2 + 2\,GA^2.$$
Subtract equations (*a*) & (*b*),
thus
$$AC^2 - AB^2 = 4\,GB.\,GD = 2\,BC.\,GD.$$

NOTE.—The third possible case, in which D coincides with G, is left as an exercise to the student.

COROLLARY 1. [L.7].—**The locus of a point, which moves so that the sum of the squares on its distances from two fixed points is constant, is a circle.**

In Fig. 379 or 380 suppose that B and C are fixed, and that A moves so that the sum of the squares on AB, AC is equal to some constant area S.

Then by T.10, $GB^2 + GA^2 = \frac{1}{2}S$,

$$i.e.\ GA^2 = \tfrac{1}{2}S - GB^2 = \text{constant},$$

for G and B are fixed points.

Thus GA is of constant length,
whence the locus of A is a circle whose centre is G.

COROLLARY 2. [L.8].—**If a point moves so that the square on its distance from one fixed point exceeds the square on its distance from another by a constant area, then the locus of the moving point is a line perpendicular to the join of the fixed points.**

In Fig. 379 or 380 suppose that B and C are fixed and that A moves so that the square on AC exceeds the square on AB by some constant area S.

Then by T.10, $BC.GD = \frac{1}{2}S$.

But BC is constant, $\therefore$ GD is a constant length, and hence D is a fixed point.

Because the projection of A on BC is a fixed point,
therefore the locus of A is a line perpendicular to BC.

247. **Formulae.**—Theorems T.8, T.9, T.10 can obviously be expressed algebraically by the following formulae :—

$$c^2 = a^2 + b^2 + 2ax, \qquad b^2 + c^2 = 2m^2 + \tfrac{1}{2}a^2,$$
$$c^2 = a^2 + b^2 - 2ax, \qquad b^2 - c^2 = 2ah.$$

where a, b, c represent the lengths of the sides of any triangle, x is the projection of the side b upon the side a, m is the median which bisects the side a, and h is the projection of this median on the side a.

These formulae can be used in many types of calculation.

Example.—*The sides of a triangle measure* 3 *cms.*, 5 *cms.*, 7 *cms. Find the length of the projection of the first side upon the second.*

In the above formulae, x is the projection of b upon a ; the question requires the projection of the side 3 cms. on the side 5 cms. Thus we put $b = 3$, $a = 5$, $c = 7$, in one of the first two formulae—we need not determine which—and solve for x.

Then $49 = 25 + 9 \pm 10x$. Obviously the positive sign is correct, which indicates that the sides 3 cms. and 5 cms. contain an *obtuse* angle. We then have $10x = 15$, $x = 1{\cdot}5$.

EXERCISES XCI.

CALCULATIONS.

1. In △ ABC, $AC = 20''$, $BC = 30''$, projection of AC on $BC = 5''$. Calculate the length of AB (i) if ∠ C is acute, (ii) if ∠ C is obtuse.

2. In △ PQR, $PQ = 4$ miles, $QR = 5$ miles, $RP = 6$ miles. Calculate the length of the projection of PR (i) on QR and (ii) on PQ.

3. The sides of a triangle measure 2 yds., 3 yds., and 4 yds. respectively. Taking the two sides which include the largest angle, calculate the projection of each of these two sides on the other.

4. In the triangle of Question 3, take the two sides which include the smallest angle. Calculate the projection of each of these two sides on the other.

5. The base of a triangle measures 7 ft., and its sides measure 5 ft. and 6 ft. Calculate the projection of the shorter side on the base. Hence calculate the altitude of the triangle, and hence its area.

6. The base of a triangle measures 4 ft. and its sides measure 6 ft. and 3 ft. Calculate the projection of the shorter side on the base. Hence calculate the altitude of the triangle, and hence its area.

7. In the triangle of Question 2, calculate the length of the median which bisects PQ.

8. In △ ABC, $AB = 3''$, $AC = 4''$, and the median $AG = 3''$. Calculate the length BC.

9. A, B, C, D are four points in a straight line, and $AB = BC = CD = 2''$. P is a point without the line such that $PA = 4''$, $PB = 3''$. Calculate the lengths PC, PD.

10. Two adjacent sides of a parallelogram measure 3 ft. and 4 ft., and one diagonal measures 6 ft. Calculate the length of the other diagonal.

RIDERS.

11. Two right-angled triangles ABC, ABD are drawn on the same side of the same hypotenuse AB. If the sides BC, AD intersect internally at O, prove that $AO.OD = BO.OC$. T.8.

12. $ABCD$ is a rectangle and P any point in its plane. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

13. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

14. The sum of the squares on the sides of any quadrilateral exceeds the sum of the squares on the diagonals by four times the square on the line joining the middle points of the diagonals. T.10.

15. If the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral is a parallelogram (see Question 14).

16. State what result is obtained from Theorem T.10—

(*a*) When $\angle A$ becomes more and more obtuse, the point A moving nearer to BC and ultimately lying in BC.

(*b*) When $\angle A$ becomes more and more acute, and $\angle B$ more and more obtuse, the point A ultimately lying in CB produced.

248. The remainder of this chapter deals with certain rectangle-theorems which can be established for chords and tangents of circles.

EXERCISES XCII.

In these Exercises teacher and class may work together.

1. Draw a circle of radius 2″. Take a point P within the circle (say about 1·2″ from the centre). Through P draw three different chords. Measure the segments of each chord and in each case calculate the area of the rectangle contained by the two segments. What do you infer?

2. Repeat Question 1, taking the point P (i) nearer the ○^{ce}, (ii) nearer the centre. What do you infer from your results?

3. A point P is taken, within a circle of centre O and radius r, at a distance d from the centre. A chord is drawn through P at right angles to OP. Prove that the area of the rectangle contained by the segments of this chord is $r^2 - d^2$.

4. In Question 3 a chord is drawn passing through P and through the centre. Prove that the area of the rectangle contained by the segments into which this chord is divided at P is $r^2 - d^2$.

What do you infer from Questions 1, 2, 3, 4?

5. Draw a circle of radius 1″. Take a point P without the circle (say about 1·6″ from the centre). Draw three different chords passing through P. Each chord is divided externally into two segments at P. In each case calculate the area of the rectangle contained by the two segments. What do you infer?

6. A point P is taken without a circle of radius r at a distance d from the centre. A chord is drawn passing through P and through the centre. Prove that the area of the rectangle contained by the segments into which this chord is divided at P is $d^2 - r^2$.

What do you infer from Questions 5, 6?

7. In Question 6 a tangent PT is drawn from P to the circle. Prove that the area of the square on the tangent is $d^2 - r^2$.

What do you infer from Questions 5, 6, 7?

8.* Show that the result of Question 7 can be inferred from the *result* of Question 5.

249. *Prel. Ex.*—In Fig. 381 prove that (i) $OB^2 - OP^2 = MB^2 - MP^2$, and (ii) $AP.\ PB = MB^2 - MP^2$.

THEOREM [RC.1].—**If two chords of a circle intersect at a point within the circumference, then the rectangle contained by the segments of the one chord shall be equal to the rectangle contained by the segments of the other.**

In Fig. 381, chords AB, CD intersect at the point P:—
Required to prove that $AP.PB = CP.PD$.

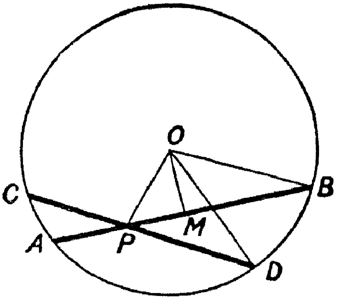

FIG. 381.

Cons.—Let O be the centre of the circle.
Join OB, OD, OP. Draw $OM \perp AB$.

Proof.— $\because$ $OM \perp$ chord AB, $\therefore$ M bisects AB. Ch. 1.
$\because$ $OM \perp PB$, $\therefore OB^2 = MB^2 + MO^2$, RT. 2.
and $OP^2 = MP^2 + MO^2$. RT. 2.

$$\begin{aligned} \text{Subtracting, } OB^2 - OP^2 &= MB^2 - MP^2 \\ &= (MB - MP)(MB + MP) \quad \text{R. 5.} \\ &= (MA - MP)(MB + MP) \end{aligned}$$

i. e. $OB^2 - OP^2 = AP.PB$. . . . . (*a*)
Similarly $OD^2 - OP^2 = CP.PD$.
But $OD = OB$, $\therefore$ $OD^2 = OB^2$. Pm. 7.
Hence $OB^2 - OP^2 = CP.PD$. . . . . (*b*)

Compare equations (*a*) and (*b*),
thus $AP.PB = CP.PD$.

COROLLARY.—**If a chord of a circle is divided internally at any point, then the rectangle contained by the segments of the chord is equal to the difference between the squares on the radius and on the line joining this point to the centre.**

250. THEOREM [RC.1*c*].—**If two lines divide each other internally so that the rectangle contained by the segments of the one line is equal to that contained by the segments of the other, then the four extremities of the two lines are concyclic points.**

In Fig. 382, the two lines AB, CD divide each other internally at the point P:—

Given that $AP.PB = CP.PD$,

Required to prove that A, B, C, D are concyclic points.

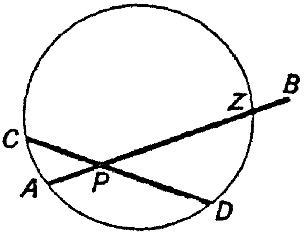

FIG. 382.

Cons.—A circle can be drawn through the three points A, C, D. Ch.3.

Let this circle meet AP a second time at the point Z.

Proof.—In ⊙ CAD, ∵ chords AZ, CD intersect at P,

$$\therefore AP.PZ = CP.PD. \qquad \text{RC.1.}$$

But $AP.PB = CP.PD$. *Hyp.*

Thus $AP.PZ = AP.PB$,

whence $PZ = PB$.†

Also Z and B fall on the same side of P;‡ hence Z coincides with B.

Thus A, B, C, D are concyclic points.

---

† If two rectangles are equal and have one pair of equal sides, then they are congruent. Easily proved by superposition and *reductio ad absurdum*. Thus it follows that if $AP.HK = AP.LM$ then $HK = LM$, and further that if $AP^2 = AP.ST$ then $AP = ST$.

‡ Another "appeal to the figure." The underlying assumptions are that P falls within ⊙ ACD, and thus AP *when produced through* P meets the circle at some point Z.

EXERCISES XCIII.

RIDERS.

1. Two lines AB, CD divide each other internally at a point P. Given that $AP.PB=CP.PD$ prove that △s APC, DPB are equiangular: and conversely.

2. In △ ABC, the lines AD, BE, CF are respectively perpendicular to the sides BC, CA, AB; also these three perpendiculars pass through a common point P. Prove that

$$AP.PD=BP.PE=CP.PF.$$

3. If through any point in the common chord of two circles which intersect, there be drawn two other chords, one in each circle, their four extremities are concyclic.

4. Find the locus of all points which divide chords of a given circle into two segments, the rectangles contained by which are equal to a given square.

5. A chord PAQ cuts the diameter of a circle in A at an angle of $\frac{1}{2}$R, prove that AP^2+AQ^2 is constant. [R.6.

6. ABC is a triangle, obtuse-angled at A; draw a line AD through A meeting BC in D and such that $AD^2=BD.DC$.

[Draw ⊙ ABC.

CALCULATIONS.

7. In Fig. 381, $PC=2''$, $PD=3''$, $PA=1{\cdot}5''$; find PB.

8. In Fig. 381, $PA=1{\cdot}2''$, $PB=3{\cdot}3''$, $PD=2{\cdot}7''$; find CD.

9. In a circle a chord of length $3''$ cuts off a segment of altitude $1{\cdot}2''$. Find the diameter of the circle.

10. In a circle a chord of length l cuts off a segment of altitude h. Find the diameter of the circle.

11. In a circle of radius 20 cms. a chord is drawn through a point 5 cms. from the centre. Find the area of the rectangle contained by the segments of this chord.

12. P is a point in a chord AB of a circle whose centre is O. Given that $PA=1$ cm., $PB=2$ cms., $OP=3$ cms., find OA.

251. *Prel. Ex.*—In Fig. 383 prove that (i) $OP^2-OB^2=MP^2-MB^2$, and (ii) $PA.PB=MP^2-MB^2$.

THEOREM [RC.2].—**If two chords of a circle intersect at a point without the circumference, then the rectangle contained by the segments of the one chord shall be equal to the rectangle contained by the segments of the other.**

In Fig. 383, chords AB, CD intersect at the point P:—
Required to prove that $PA.PB=PC.PD$.

Cons.—Let O be the centre of the circle.
Join OB, OD, OP. Draw $OM \perp AB$.

Proof.—$\because OM \perp$ chord AB, $\therefore M$ bisects AB. Ch.1.

$\because OM \perp PB$, $\therefore OP^2 = MP^2 + MO^2$ RT.2.

and $OB^2 = MB^2 + MO^2$. RT.2.

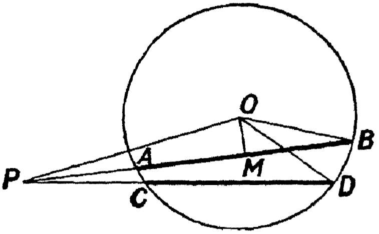

FIG. 383.

Subtracting, $OP^2 - OB^2 = MP^2 - MB^2$

$= (MP - MB)(MP + MB)$ R.5.

$= (MP - MA)(MP + MB)$,

i. e. $OP^2 - OB^2 = PA.PB$. . . . . (*a*)

Similarly $OP^2 - OD^2 = PC.PD$.

But $OD = OB$, $\therefore OD^2 = OB^2$. Pm.7.

Hence $OP^2 - OB^2 = PC.PD$. . . . . (*b*)

Compare equations (*a*) and (*b*),

thus $PA.PB = PC.PD$.

COROLLARY 1.—**If a chord of a circle is divided externally at any point, then the rectangle contained by the segments of the chord is equal to the difference between the square on the radius and the square on the line joining the point to the centre.**

COROLLARY 2. [RC.3]*.—**If a chord of a circle is divided externally by a tangent, then the square on the tangent is equal to the rectangle contained by the segments of the chord.**

In Fig 383, suppose that the secant PCD swings round P until the points C and D coincide.

Then PC is a tangent to $\odot ABC$. Also $PC.PD = PC^2$.

But $PC.PD = PA.PB$; hence $PC^2 = PA.PB$.

NOTE.—From Theorems RC.1 and RC.2 we obtain the result that $PA.PB =$ the difference between OB^2 and OP^2 $(= OB^2 \sim OP^2)$ whether P lies within or without the circle (Figs. 381, 383).

252. THEOREM [RC.3].—**If a chord of a circle is divided externally by a tangent, the rectangle contained by the segments of the chord is equal to the square on the tangent.**

In Fig. 384, tangent PT intersects chord AB at P.
Required to prove that $PB = PT^2$.

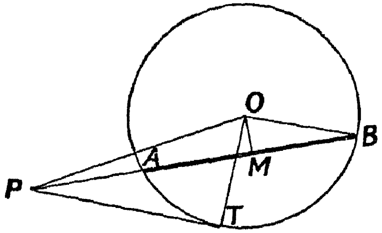

FIG. 384.

Cons.—Let O be the centre of the circle.
Join OB, OT, OP. Draw $OM \perp AB$.

Proof.—$\because OM \perp$ chord AB, $\therefore M$ bisects AB. Ch. 1.
$\because OM \perp PB$, $\therefore OP^2 = MP^2 + MO^2$, RT. 2.
and $OB^2 = MB^2 + MO^2$. RT. 2.

$$\text{Subtracting,}\quad OP^2 - OB^2 = MP^2 - MB^2$$
$$= (MP - MB)(MP + MB) \qquad \text{R. 5.}$$
$$= (MP - MA)(MP + MB)$$
$$\text{i.e.}\quad OP^2 - OB^2 = PA.PB \ldots\ldots\ldots\ldots (a)$$

$\because \angle OTP = \text{R}$ (Tn. 1), $\therefore OP^2 - OT^2 = PT^2$. RT. 2.
But $OT = OB$, $\therefore OT^2 = OB^2$. Pm. 7.
Hence $OP^2 - OB^2 = PT^2 \ldots\ldots\ldots\ldots (b)$
Compare equations (a) and (b); thus $PA.PB = PT^2$.

253. THEOREM [RC.2c].—**If two lines divide each other externally so that the rectangle contained by the segments of the one line is equal to that contained by the segments of the other, the four extremities of the two lines are concyclic points.**

In Fig. 385, the two lines AB, CD divide each other externally at P;—
Given that $PA.PB = PC.PD$,
Required to prove that A, B, C, D are concyclic.
Cons.—Draw a circle through the points A, C, D. Ch. 3.

Proof.—PA is not a tangent to $\odot\ ACD$; for if it were we should have $PA^2 = PC.PD$, which is not the case.

Hence PA meets the $\bigcirc^{ce}$ at some second point Z.

In $\odot\ ACD$, $\because$ chords AZ, CD intersect at P,

$\therefore\ PA.PZ = PC.PD$. RC.2.

But $PA.PB = PC.PD$. *Hyp.*

Thus $PA.PZ = PA.PB$; whence $PZ = PB$.[†]

Also Z and B lie on the same side of P,

hence Z coincides with B.

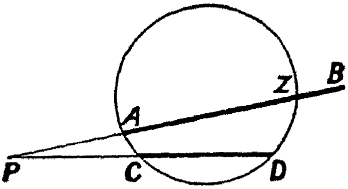

FIG. 385.

Thus A, B, C, D are concyclic points.

254. THEOREM [RC.3*c*].—**If a chord of a circle is divided externally at a given point and if a line is drawn from this point to the circumference such that the square on this line is equal to the rectangle contained by the segments of the chord, then this line touches the circle.**

In Fig. 385 :—

Given that $PA.PB = PT^2$,

Required to prove that PT is a tangent to the circle.

Proof.—If possible, suppose that PT is not a tangent to $\odot\ ABT$; then PT will meet $\bigcirc^{ce}$ again at some point Z.

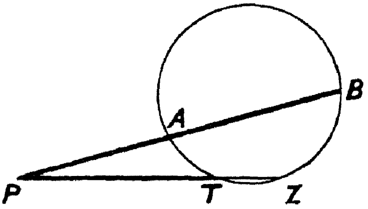

FIG. 386.

$\because$ chords AB, TZ meet at the point P,

$\therefore\ PA.PB = PT.PZ$. RC.2.

But $PA.PB = PT^2$. *Hyp.*

Hence $PT.PZ = PT^2$, and therefore $PZ = PT$.[†]

But this is not the case, for Z and T are on the same side of P. **Hence** PT must be a tangent to $\odot\ ABT$.

[†] See footnote on p. 359.

EXERCISES XCIV.

RIDERS.

1. If two circles intersect, then the common chord produced bisects each of the common tangents.

2. Two lines *AB, CD* divide each other externally at a point *P.* Given that $PA.PB = PC.PD$ prove that △s *APC, DPB* are equiangular: and conversely.

3. From a point *P* are drawn two straight lines *PAB* and *PT.* Given that $PA.PB = PT^2$ prove that △s *APT, TPB* are equiangular: and conversely.

4. In △*ABC,* the lines *AD, BE, CF* are respectively perpendicular to the sides *BC, CA, AB;* also these three perpendiculars pass through a common point *P.* Prove that $AF.AB = AP.AD = AC.AE$.

Write down two other similar results in the same figure.

5. Tangents are drawn to two intersecting circles from a point in their common chord produced. Prove that these tangents are of equal length.

6. Two chords of two intersecting circles meet on the common chord produced. Prove that the four extremities of these two chords are concyclic.

7. Two circles touch one another, and two chords of these circles meet on the tangent at the point of contact. Prove that the four extremities of these two chords are concyclic.

CALCULATIONS.

8. In Fig. 383, $PA = 1''$, $AB = 2''$, $PC = 1{\cdot}2''$; find *CD.*

9. In Fig. 384, $PA = 1{\cdot}2''$, $AB = 2{\cdot}3''$; find *PT.*

10. In a circle of radius 12 cms. a chord is drawn. Find the area of the rectangle contained by the segments into which this chord is divided by a point 20 cms. from the centre.

11. In Fig. 383, given that $PA =$ 2 cms., $AB =$ 3 cms., $OP =$ 5 cms., find *OA.*

255. **Multiplication, Division and Square Root.**—By means of Theorems RC.1, RC.2, RC.3 various arithmetical processes can be performed by geometrical construction. For example:—

(i) **To evaluate** $\sqrt{ab}$.—Draw a circle of convenient size (Fig. 384): draw a chord *BA* of length $(b - a)$ and produce it to *P,* making $PA = a$: draw and measure the tangent *PT.*

(ii) **To evaluate** a^2/b.—Draw a circle of convenient size (Fig. 384): draw a tangent *PT* of length a: with centre *P* and radius b draw an arc cutting ○^{ce} at *B.* Join *PB* meeting ○^{ce} again at *A:* measure *PA.*

In both cases the proof is easy.

NOTE.—A better method for determining $\sqrt{ab}$ can however be obtained from § 257. See Ques. 1–3, Exercises XCVII.

EXERCISES XCV.

PROBLEMS IN PRACTICAL GEOMETRY.

1. Use Theorem RC.1 to evaluate the following:—
(i) $13 \times 25 \div 31$, (ii) $3{\cdot}7 \div 5{\cdot}7$, (iii) $570 \div 37$, (iv) $2{\cdot}3^2 \div 3{\cdot}6$.

2. Use Theorems RC.2 and RC.3 to evaluate the following:—
(i) $7{\cdot}3 \times 3{\cdot}7 \div 4{\cdot}3$, (ii) $3{\cdot}7 \times 4{\cdot}3 \div 4{\cdot}7$, (iii) $4{\cdot}3^2 \div 5{\cdot}7$, (iv) $3{\cdot}7^2 \div 2{\cdot}9$, (v) $\sqrt{7{\cdot}3 \times 3{\cdot}7}$, (vi) $\sqrt{29 \times 43}$, (vii) $\sqrt{4{\cdot}7}$, (viii) $\sqrt{137}$.

CALCULATIONS.

3. Find the sum of the areas of the squares on the medians of a triangle whose sides measure a, b, c.

4. The sides of a triangle measure 3, 5, 7 ft. Calculate its area.

5. The medians of a triangle measure 2″, 3″, 4″: find the sides.

6. PT is a tangent to a circle and PAB a secant. Given $\angle P = \text{R}$, $PT = t$, $PA = a$, find the radius of the circle.

7. PT is a tangent to a circle and PAB a secant. Given $\angle P = \text{R}$, $PA = 2a$, $PB = 2b$, find the distance of P from the centre.

8. Calculate the diameter of a circle which circumscribes a triangle whose sides measure 5, 5, 6 ft.

9. Calculate the diameter of a circle which circumscribes an equilateral triangle of side 2″.

10. AB, CD are two perpendicular chords of a circle which intersect at a point P within the circle. Given that $PA = 1''$, $PB = 4''$, $PC = {\cdot}8''$, calculate PD and the radius of the circle.

11. AB, CD are two perpendicular chords of a circle which intersect at a point P without the circle. Given that $PA = 1''$, $PB = 4''$, $PC = {\cdot}8''$, calculate PD and the radius of the circle.

RIDERS.

12. The chords of two intersecting circles which are bisected at any point of the common chord are equal.

13. If the tangents from a given point to any number of intersecting circles are equal, all the common chords of the circles pass through that point.

14. Circles are described passing through two fixed points; find the locus of a point from which the tangents to all the circles are equal.

15. The square on the straight line drawn from the vertex of an isosceles triangle to any point in the base is less than the square on either of the equal sides by the rectangle contained by the segments of the base.

16. In $\triangle ABC$, $\angle C = \frac{4}{3}\text{R}$; prove that $AB^2 = CA^2 + CB^2 + CA.CB$.

17. ABC is a triangle having the angle A a right angle; prove that four times the sum of the squares on the medians BE, CF is equal to five times the square on the hypotenuse BC.

18. E is a point in the base BC of a triangle ABC such that $BE = 2\,EC$. Prove that $AB^2 + 2\,AC^2 = 6\,EC^2 + 3\,AE^2$. T.8, T.9.

19.* F is a point in the base BC of a triangle ABC such that $m\,FB = n\,FC$. Prove that

$$m\,AB^2 + n\,AC^2 = m\,FB^2 + n\,FC^2 + (m+n)\,AF^2. \qquad \text{T.8, T.9.}$$

20*. If the diagonals of a parallelogram $ABCD$ intersect in G, and if P be any point, prove that the sum of the squares on PA, PB, PC, PD is equal to the sum of the squares on GA, GB, GC, GD, together with four times the square on PG.

21.* If A, B, C, D are fixed points, and P is a point such that the sum of the squares on PA, PB, PC, PD is constant, prove that the locus of P is a circle, whose centre is the point of intersection of the straight lines joining the middle points of AB, CD and of AD, BC.

*THE RADICAL AXIS.**

256. DEFINITION.—**The radical axis of two given circles is the locus of a point from which the tangents to the two circles are of equal length.**

Thus in Fig. 387 if P moves so that tangent PC = tangent PD, then the locus of P is the radical axis.

THEOREM.*—**To prove that the radical axis of two circles is a straight line perpendicular to the join of their centres.**

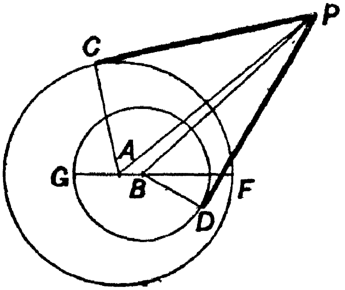

FIG. 387.

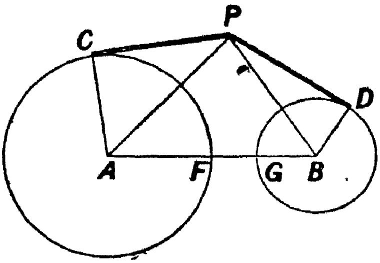

FIG. 388.

In Figs. 387, 388 PC and PD are the tangents from P to the circles whose centres are A, B.

Given that $PC = PD$.

Required to prove that locus of P is a st. line $\perp AB$.

Proof.—$\because PC = PD, \therefore PC^2 = PD^2$: Pm.7.

i.e. $PA^2 - AC^2 = PB^2 - BD^2$, RT.2.

$\therefore PA^2 - PB^2 = AC^2 - BD^2 = AF^2 - BG^2$.

Thus $PA^2 - PB^2$ is constant;

hence the locus of P is a st. line $\perp AB$. L.8.

PROBLEM.*—**To construct the radical axis of two given circles.**

In Fig. 389 :—

Required to construct the radical axis of the two circles GQH, KLR.

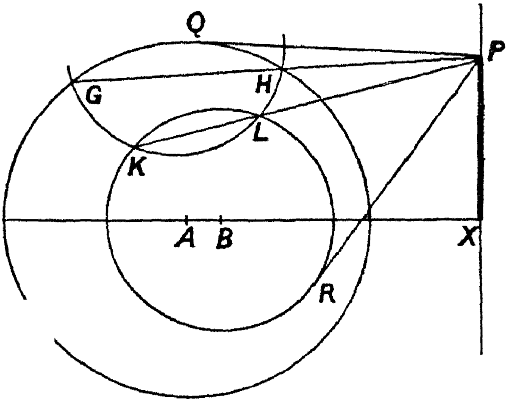

FIG. 389.

Cons.—With any convenient centre and radius describe a circle GKL cutting the two given circles at G, H, and K, L respectively.

Let the chords GH and KL intersect at P.

Through P draw PX perpendicular to the join of the centres A, B.

Then PX is the required radical axis.

Proof.—Let PQ, PR be tangents to the two given ⊙s.

In ⊙ GQH, $PQ^2 = PG.\ PH$. RC. 3.
In ⊙ GKL, $PG.PH = PK.\ PL$. RC. 2.
In ⊙ KRL, $PK.PL = PR^2$. RC. 3.

Thus $PQ^2 = PR^2$, whence $PQ = PR$.
∴ P is a point on the radical axis.

But the radical axis is a st. line ⊥ AB. *Proved.*

Hence PX is the required radical axis.

NOTE.—If the two circles touch or cut the above construction may be replaced by the simpler constructions in the next two riders.

*EXERCISES XCVI** (*Riders*).

1. **If two circles touch one another internally or externally their radical axis is the common tangent at their point of contact.**

2. **If two circles cut one another their radical axis is their common chord produced.**†

3. **Given three circles X, Y, Z, whose centres are not collinear, prove that the radical axes of the three pairs of circles (X, Y), (Y, Z), (Z, X) are concurrent.**

DEFINITION.—The **radical centre** of three circles is the point of intersection of their three radical axes.

4. Show how to divide a given line so that the difference on the squares of the segments is equal to a given square.

5. The centres of two circles are A, B, and AB cuts the circumferences at H, K respectively; also their radical axis meets AB at X. If radius $AH >$ radius BK, prove that $AX^2 - BX^2 = AH^2 - BK^2$.

6. In the figure of Ques. 5, prove the following construction for the radical axis:—

Draw $BM \perp AB$; with centre K and radius AH describe an arc cutting BM at M. Bisect AM at right angles by a line cutting AB at X. The line through $X \perp AB$ shall be the radical axis.

7. Two circles do not intersect: show that the middle points of their four common tangents are collinear.

8. Find the locus of the centre of a circle which cuts two given circles orthogonally.

9. Show how to draw a circle to cut each of three given circles orthogonally.

10. P is a moving point, and A a fixed point; also PT is the tangent from P to a fixed circle. Find the locus of P if $PT = PA$.

11. A, B, C, D are four points on a straight line. Find a point O on this line such that $OA.OD = OB.OC$.

12. Find the locus of a point P, if the difference of the squares on the tangents from P to two given circles is constant.

---

† If H, K are the points of intersection, then according to the definition given above the radical axis includes only the *produced portions* of HK. In advanced mathematics however the radical axis is reckoned to include the whole line; the tangents from a point between H and K are *imaginary* and equal.

CHAPTER XVII.

PROBLEMS ON RECTANGLES.

257. PROBLEM.—**To construct a square equal to a given rectangle.**

In Fig. 390 :—

Required to construct a square equal to rect. $ABCD$.

Cons.—Produce AB to E, making $BE = BC$.

On diam. AE describe a semi-circle.

Draw $BG \perp AE$, to meet $\bigcirc^{ce}$ at G.

Then will $BG^2 =$ rect. AC.

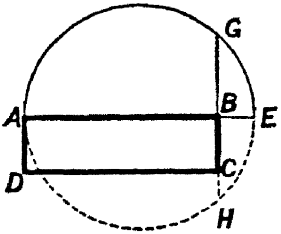

FIG. 390.

Proof.—Let GB meet $\bigcirc^{ce}$ again at H.

$\because$ diam. $AE \perp$ chord GH,

$\therefore BG = BH$. Ch.1.

$\because$ chords AE, GH intersect at B,

$\therefore GB.BH = AB.BE$; RC.1.

that is $GB^2 =$ rect. AC.

258. PROBLEM.—**To divide a given line internally so that the rectangle contained by the segments is equal to the square on a given line.**

In Fig. 391 :—

Required to divide AB internally so that the rect. contained by the segments shall be equal to the sq. on X.

Cons.—On diam. AB describe a semi-circle.

Draw $BD \perp AB$ and equal to X.

Draw $DE \parallel AB$ to meet $\bigcirc^{ce}$ at E. Draw $EF \perp AB$.

Then $AF.FB = X^2$.

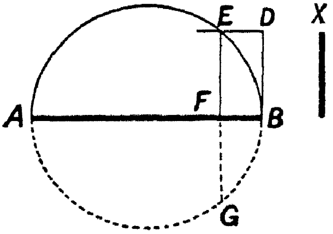

FIG. 391.

Proof.—$\because EF \parallel DB$ (P.2) and $ED \parallel FB$, $\therefore EF = DB$. Pm.1.

Let EF meet $\bigcirc^{ce}$ again at G.

$\because$ diam. $AB \perp$ chord EG, $\therefore EF = FG$. Ch.1.

$\because$ chords AB, EG intersect at F,

$\therefore AF.FB = EF.FG = EF^2 = DB^2$. RC.1.

that is $AF.FB = X^2$.

259. *Prel. Ex.*—(i) In Fig 392 E is the centre of the arc DF and bisects the line AB; also $BD \perp AB$. Show that $AF.FB = BD^2$. (ii) Hence show how to divide AB externally so that the rectangle contained by the segments is equal to the square on C.

PROBLEM.—**To divide a given line externally so that the rectangle contained by the segments is equal to the square on a given line.**

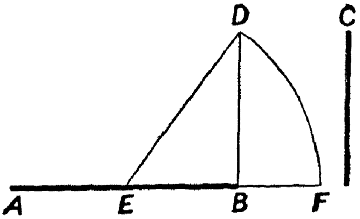

FIG. 392.

In Fig. 392:—

Required to divide AB externally so that the rect. contained by the segments shall be equal to the sq. on C.

Cons.—Bisect AB at E. Draw $BD \perp AB$ and $= C$. With centre E and radius ED describe an arc meeting AB produced at F.

Then will $AF.FB = C^2$.

Proof.—
$$C^2 = BD^2 = ED^2 - EB^2 \qquad \text{RT.2.}$$
$$= (ED + EB)(ED - EB) \qquad \text{R.5.}$$
$$= (EF + EA)(EF - EB)$$
that is $$C^2 = AF.FB.$$

EXERCISES XCVII.

PRACTICAL GEOMETRY.

1. Adjacent sides of a rectangle measures 3 cms. and 2 cms. Construct a square of equal area, and measure its side. Verify by calculation.

2. Adjacent sides of a rectangle measure 1·5″ and ·6″. Find by construction and measurement the side of a square of equal area. Verify by calculation.

3. Give a geometrical construction for determining the value of $\sqrt{ab}$ where a and b are given numbers? § 257.

4. Evaluate geometrically (i) $\sqrt{5}$; (ii) $\sqrt{10}$; (iii) $\sqrt{21}$; (iv) $\sqrt{35}$; (v) $\sqrt{70}$; (vi) $\sqrt{\cdot 6}$; (vii) $\sqrt{2\cdot 3}$; (viii) $\sqrt{\cdot 75}$.

5. Divide a line of length 7 cms. (i) internally, (ii) externally, so that in each case the rectangle contained by the segments shall have an area of 9 sq. cms. Measure the segments.

6. Divide a line of length 2″ (i) internally, (ii) externally, so that in each case the rectangle contained by the segments shall have an area of ·49 sq. in. Measure the segments.

7. Give a geometrical construction for finding two numbers whose sum is a and product b^2. § 258.

8. Find geometrically two numbers whose sum is 9 and whose product is (i) 9, (ii) 4, (iii) $2\frac{1}{4}$.

Hence solve the quadratic equations :—

(iv) $x^2-9x+9=0$, (v) $x^2+9x+4=0$, (vi) $4x^2-36x+9=0$.

9. Find geometrically two numbers whose sum is 20 and whose product is (i) 100, (ii) 49, (iii) 70.

10. What is the greatest possible value of the product of two numbers whose sum is (i) 6, (ii) 7, (iii) x?

11. What is the least possible value of the sum of two numbers whose product is (i) 9, (ii) $2\frac{1}{4}$, (iii) x?

12. Give a geometrical construction for finding two numbers whose difference is a and product b^2. § 259.

13. Find geometrically two numbers whose difference is 5 and whose product is (i) 9, (ii) 4, (iii) $2\frac{1}{4}$.

Hence solve the quadratic equations :—

(iv) $x^2-5x-9=0$, (v) $x^2+5x-4=0$, (vi) $4x^2-20x-9=0$.

14. Find geometrically two numbers whose difference is 20 and whose product is (i) 100, (ii) 49, (iii) 70.

260. On Medial Section :—

DEFINITION.—**If a given line is divided at a point into two segments such that the square on one segment is equal to the rectangle contained by the given line and the other segment, then the line is said to be divided in** *medial section* **(or in** *extreme and mean ratio*).

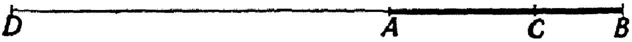

FIG. 393.

A line may be divided either internally or externally in medial section. Thus in Fig. 393 the line AB is divided internally in medial section at C, and is divided externally in medial section at D.

In other words

$$(\text{segment } AC)^2 = \text{line } AB.\ \text{segment } BC,$$
$$\text{and } (\text{segment } AD)^2 = \text{line } AB.\ \text{segment } BD.$$

Verify these statements by measurement and calculation.

261. *Prel. Ex.*—In Fig. 394 $AB = L$, $AC = X$, $BD = \frac{1}{2}L$, $BD \perp AB$, D is centre of arc EBF, A is centre of arc EC. Prove that

(i) $L^2 = X(X+L)$, RC.3.

and hence that (ii) $X^2 = L(L-X)$. R.2.

PROBLEM.—**To divide a given line internally in medial section.**

In Fig. 394 :—*Required* to divide AB in medial section.

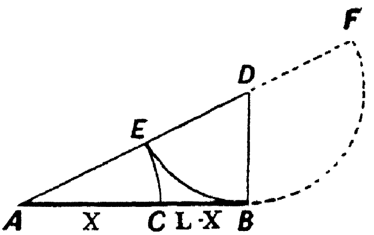

FIG. 394.

Cons.—Draw $BD \perp AB$, and equal to $\frac{1}{2}$ AB. Join AD.

With centre D describe arc BE meeting AD at E.

With centre A describe arc EC meeting AB at C.

Then C divides AB internally in medial section; that is $AC^2 = AB.BC$.

Proof.†—Let arc EB meet AD produced at F. Let $AB = L$, $AC = X$.
Then $DB = \frac{1}{2}L$, $EF = L$, $AE = X$, $AF = X + L$, $BC = L - X$.
Now in ⊙ EBF, $AB \perp$ radius DB; ∴ AB touches ⊙ EBF. Tn.1.
Hence $AE.AF = AB^2$; *i. e.* $X(X+L) = L^2$. RC.3.

$$\therefore\ X^2 + X.L = L^2; \qquad \text{R.2.}$$
$$\therefore\ X^2 = L^2 - L.X$$
$$= L\,(L - X). \qquad \text{R.2.}$$

that is $AC^2 = AB.BC.$

262. *Prel. Ex.*—In Fig. 395 $AB = L$, $AC = X$, $BD = \frac{1}{2}L$, $BD \perp AB$, D is centre of arc EBF, A is centre of arc EC. Prove that

(i) $L^2 = X\,(X - L)$, RC.3.

and hence that (ii) $X^2 = L\,(L + X)$. R.2.

PROBLEM.—**To divide a given line externally in medial section.**

In Fig. 395 :—

Required to divide AB externally in medial section.

Cons.—Draw $BD \perp AB$, and equal to $\frac{1}{2}AB$. Join AD.

With centre D and radius DB describe arc BE meeting AD produced at E. With centre A and radius AE describe arc EC, meeting BA produced at C.

Then C divides AB externally in medial section; that is $AC^2 = AB.BC$.

---

† This proof is due to Mr. R. F. Davis, by whose kind permission it is inserted here.

Proof.—Let arc EB meet AD at F. Let $AB = L$, $AC = X$.
Then $BD = \frac{1}{2} L$, $EF = L$, $AE = X$, $AF = X - L$, $BC = X + L$.

Now in ⊙ EBF, $AB \perp$ radius DB; $\therefore$ AB touches ⊙ EBF. Tn.1.
Hence $AE.AF = AB^2$; *i. e.* $X(X-L) = L^2$. RC.3.

$$\therefore X^2 - X.L = L^2. \qquad \text{R.2.}$$
$$\therefore X^2 = L^2 + L.X$$
$$= L(L+X). \qquad \text{R.2.}$$

that is
$$AC^2 = AB.BC.$$

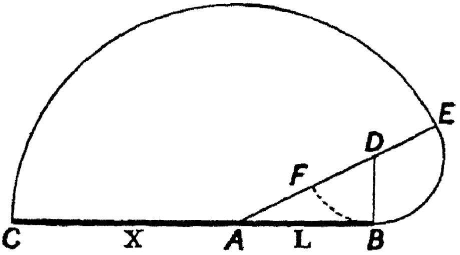

FIG. 395.

EXERCISES XCVIII.

PRACTICAL GEOMETRY.

1. Take a line AB, 8 cms. long. Divide it internally in medial section at C (AC to be the larger segment) and measure the segments. Verify the accuracy of your work by calculation.

2. Construct triangles having their sides equal to the following lines in Question 1 :—

(i) AB, AB, AC; (ii) AC, AC, BC; (iii) AC, AC, AB. Measure the angles of each triangle.

In each triangle *what simple relation do you notice between the angles?*

3. Take a line AB, 3 cms. long. Divide it externally in extreme and mean ratio at C (AC to be the smaller segment) and measure the segments. Verify the accuracy of your work by calculation.

4. Repeat Question 2, taking the lengths from Question 3 instead of from Question 1.

CALCULATIONS.

5. A line of length 1″ is divided internally in medial section. Calculate the lengths of the segments algebraically.

6. A line of length a is divided internally in medial section. Calculate the lengths of the segments algebraically, and explain the double answer.

RIDERS.

7. AB is divided internally in medial section at C. Prove that BC, the smaller segment, is divided externally in medial section at A.

8. AB is divided internally in medial section at C, and from the larger segment AC a portion AD is cut off equal to BC. Prove that AC is divided internally in medial section. R.2.

263. PROBLEM.—**To construct an isosceles triangle having each base angle double of the vertical angle, given one side of the triangle.**

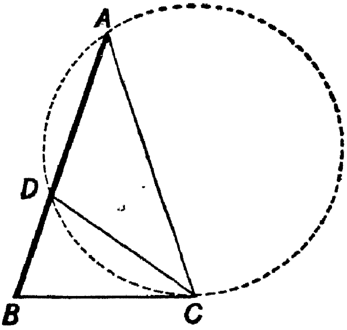

FIG. 396.

In Fig. 396 :—

Required to describe an isosceles triangle having AB for one of the equal sides, and having each base angle double of the vertical angle.

Cons.—Divide AB internally in medial section at D, AD being the larger segment. § 261.

Describe $\triangle\ BDC$ having $BC = CD = DA$. Join AC.

Then $\triangle\ ABC$ is the required triangle; that is $\angle B = \angle C = 2\angle A$.

Proof.—Describe a circle through A, D, C.
Now D divides AB in medial section; *Cons.*

$$i.e.\ AD^2 = BD.BA.$$
$$\therefore\ BC^2 = BD.BA.$$

Hence BC is a tangent to $\odot\ ADC$. RC.3c.
Also CD is a chord of the same circle; $\therefore\ \angle BCD = \angle A$. AC.5.
In $\triangle\ DAC$, $\because\ DA = DC$, $\therefore\ \angle DCA = \angle A$. T.2.
Hence $\angle BCA = 2\angle A$.
In $\triangle\ DAC$, ext. $\angle BDC = \angle A + \angle DCA = 2\angle A$. T.1.
In $\triangle\ CBD$, $\because\ CD = CB$, $\therefore\ \angle B = \angle BDC$. T.2.
Hence $\angle B = 2\angle A$.

COROLLARY.—**To describe an angle equal to $\frac{1}{5}$R.**

Cons.—In Fig. 396 bisect $\angle A$. Then $\frac{1}{2}\angle A = \frac{1}{5}$ R.

Proof.—For $\angle A + \angle B + \angle C = 2$ R, T.1.
$\therefore\ \angle A + 2\angle A + 2\angle A = 2$ R,
$\therefore\ 5\angle A = 2$ R, whence $\frac{1}{2}\angle A = \frac{1}{5}$R.

264.† PROBLEM.—**To construct an isosceles triangle having each base angle double of the vertical angle, given the length of the base.**

In Fig. 397 :—

Required to construct an isosceles triangle having its base equal to DE, and having each base angle double of the vertical angle.

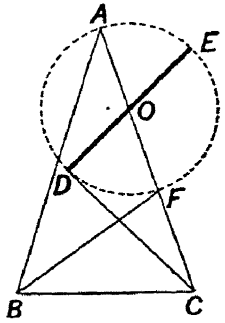

FIG. 397.

Cons.—On diam. DE describe a circle.
At D draw tangent $DC = DE$.
Through C draw secant CFA passing through centre O.
Describe $\triangle CFB$ having $CB = FB = DE$. Join BA.
Then $\triangle ABC$ is required triangle;

$$i.e.\ \angle B = \angle C = 2\angle A.$$

Proof.— $\because CD$ touches $\odot ADF$, $\therefore CD^2 = CF.CA$. RC.3.
But $CD = DE = CB$; $\therefore CB^2 = CF.CA$.
Hence CB is tangent to $\odot ABF$. RC.3*c*.
Also BF is a chord of the same circle; $\therefore \angle CBF = \angle A$. AC.5.
In $\triangle ABF$, $\because FB = DE = FA$, $\therefore \angle FBA = \angle A$. T.2.
Hence $\angle CBA = 2\angle A$.
In $\triangle ABF$, ext. $\angle BFC = \angle A + \angle FBA = 2\angle A$. T.1.
In $\triangle BFC$, $\because BF = BC$, $\therefore \angle BCF = \angle BFC$. T.2.
Hence $\angle BCA = 2\angle A$.

---

† This construction is due to Mr. R. F. Davis, by whose kind permission it is inserted here.

EXERCISES XCIX.

PRACTICAL.

1. On a base of length 1·6″ describe an isosceles triangle having each base angle double of the vertical angle. Find its area.

2. Take a line *AB* of length 6 cms. Divide it internally in medial section at *C*—*AC* being the larger segment. Construct a pentagon *PQRST* such that each side is equal to *AC* and that each of the diagonals *PR, PS* is equal to *AB.*

Measure all the angles and all the diagonals of this pentagon.

RIDERS.

3. Show that a polygon can be inscribed in a given circle each of whose sides subtends at the centre an angle equal to ∠ *B* in the figure of § 263.

How many sides has this polygon?

4. A radius of a given circle is divided internally in medial section. Show that a polygon can be inscribed in the circle each of whose sides is equal to the larger segment of the radius.

How many sides has this polygon? § 263.

5. Without using a protractor, construct an isosceles triangle having each angle at the base one-third of the vertical angle, given the length of the base.

6. Produce a given line *AB* to a point *C* such that *B* divides *AC* in medial section, and *AB* is the larger segment.

265. **Graphical Solution of Quadratic Equations.**—Any equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation. Such an equation has two and only two roots, and by a well-known theorem in algebra the sum of these roots is $-\frac{b}{a}$, and their product is $\frac{c}{a}$. Given the sum and product of the roots it is possible to solve the equation (*i. e.* to find the roots themselves) *by a geometrical construction.*† The method here given‡ was discovered by the famous writer Thomas Carlyle.

The problem is complicated by the fact that the roots may be positive or negative or even imaginary.‡‡ Consider first the following simple example in which none of these difficulties occurs.

---

† Provided that the *coefficients* are real quantities.

‡ Another method of attacking this problem has been indicated in Questions 8 and 13 of Exercises XCVII. Carlyle's method is, however, superior, as it does not involve the extraction of a square root, and it solves all quadratics (even if the roots are imaginary) by one single rule.

‡‡ Quadratics with real *coefficients* may have imaginary *roots.*

266. *Prel. Ex.*—In Fig. 398 OH and MK are ⊥ to OM, and HK is diam. of ⊙ HPK. Prove that (i) $OP + OQ = OM$, (ii) $OP.\,OQ = OH.\,MK$.

Example.—*Solve the quadratic*

$$x^2 - 5x + 3 = 0.$$

Calculation.—Here $a = 1$, $b = -5$, $c = 3$.

Thus sum of roots $= -\dfrac{b}{a} = \mathbf{5}$; product of roots $= \dfrac{c}{a} = \mathbf{3}$.

Required to find two numbers whose sum is 5 and whose product is 3.

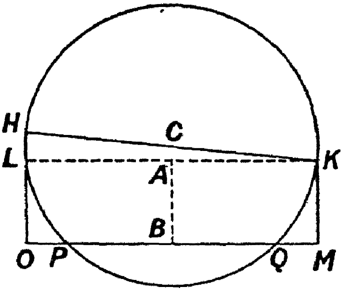

FIG. 398.

Cons.—Choose some convenient unit of length *e.g.* the inch. (In Fig. 398 the unit is $\frac{1}{4}''$.)

Draw OM to represent the *sum of the roots, i.e.*, $OM = 5$ units.

Draw two lines OH, MK, each ⊥ OM, and such that the *product* of their lengths represents the *product of the roots ; i.e.* $OH \times MK = 3$.

Say $OH = 2$ units, $MK = 1{\cdot}5$ units.‡

Draw ⊙ on diam. HK, intersecting OH in L, and OM in P and Q.

Then OP, OQ *represent the roots of the quadratic.*

Explanation.—It is easy to prove that $\angle HLK = \angle R$;

whence $LK \parallel OM$, and $OL = MK$.

Draw CAB from the centre C, ⊥ to LK and to OM. CB bisects LK and therefore OM: also CB bisects PQ. Hence $OP = QM$.

Thus $OP + OQ = QM + OQ = OM = 5$,

and $OP.OQ = OH.OL = OH.MK = 3$.

Hence OP, OQ represent the required roots of the quadratic.

By measuring OP, OQ the roots are found to be $\cdot 7$ and $\mathbf{4{\cdot}3}$.

EXERCISES C.

Solve geometrically the following quadratics :—

1. $x^2 - 6x + 4 = 0$. **2.** $x^2 - 7x + 8 = 0$. **3.** $2x^2 - 7x + 2 = 0$.

4. $5x^2 - 5x + 1 = 0$. **5.** $3x^2 - 6x + 2 = 0$. **6.** $4x^2 - 10x + 5 = 0$.

---

‡ If OH and MK differ greatly in length it may be difficult to get accurate results.

267. General Method.—The method for solving any quadratic equation is essentially the same as the method used in § 266, but certain modifications are required for dealing with negative quantities.

RULE.—*To solve any quadratic equation* construct a figure of the type of Fig. 398, such that *OM* represents the **sum of the roots,** and $OH \times MK$ represents the **product of the roots.** Let $\odot$ on diam. *HK* cut *OM* at *P, Q;* then *OP, OQ* **represent the required roots.**

To include all cases use the following **conventions as to sign:**—

(i) **Horizontal lines** represent **positive** quantities if drawn to the **right** of *O,* and **negative** if drawn to the left of *O.*

(ii) **Vertical lines** represent **positive** quantities if drawn **above** *OM,* and **negative** if drawn **below** *OM.*

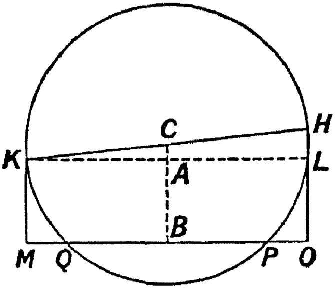

FIG. 399.

Thus in Fig. 398, *OM, OH, MK, OP, OQ* all represent positive quantities, while Figs. 399, 400, 401, show the three possible modifications of Fig. 398 produced by the conventions as to sign. Thus *OM* represents a *positive sum of roots* in Fig. 400, and a *negative sum of roots* in Figs. 399, 401. *OH* is positive in all these Figures, while *MK* is positive in Fig. 399, and negative in Figs. 400, 401: thus $OH \times MK$ represents a *positive product of roots* in Fig. 399, and a *negative product of roots* in Figs. 400, 401. Similarly *OP* represents a *positive root* in Fig. 401, and a *negative root* in Figs. 399, 400.

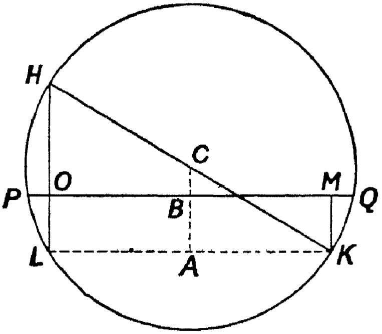

FIG. 400.

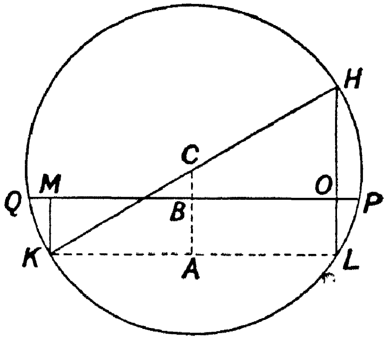

FIG. 401.

Proof.—In order to prove this method it is necessary to show that in each of these four figures

$$\text{(i)} \quad OP = MQ \text{ and } OL = MK\,[†]$$

and hence that in each figure, *if we adopt the above conventions of sign,*

$$\text{(ii)} \quad OP + OQ = OM, \quad OP \times OQ = OH \times MK.$$

Example.—*Solve the equation*

$$10x^2 + 45x - 17 = 0.$$

Here $a = 10$, $b = 45$, $c = -17$.

Thus sum of roots $= -\frac{b}{a} = -4{\cdot}5$; product of roots $= \frac{c}{a} = -1{\cdot}7$.

Since the sum of the roots is negative draw OM pointing to the *left*, and of length 4·5 cms.

Since the product of the roots is negative draw OH *upwards* and MK *downwards:* take $OH = 1{\cdot}7$ cms., $MK = 1$ cm.

Thus we obtain a figure similar to Fig. 401.

OP represents a positive root, and OQ a negative root. By measurement we obtain the answer, viz. $x = {\cdot}35$ or $-4{\cdot}85$.

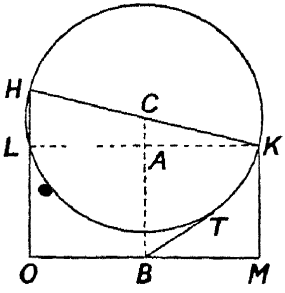

FIG. 402.

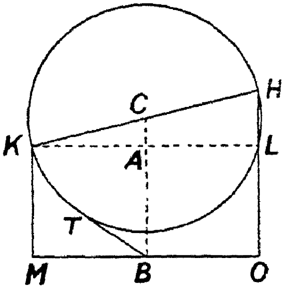

FIG. 403.

268.* Imaginary Roots.—Figs. 402, 403 represent two possible modifications of Fig. 398, in which the points P, Q disappear, as ⊙ HKL does not meet the line OM. In this case the **roots are imaginary** and are represented by

$$OB \pm i.\, BT$$

where BT is the tangent drawn from the middle point of OM, and $i. = \sqrt{-1}$.

Note that OB represents a positive or negative quantity according as it points right or left.

Proof.—The **sum** of the two quantities

$$OB + i.\, BT, \quad OB - i.\, BT$$

is obviously $\qquad 2\,OB$ or OM.

The **product** of these two quantities $= OB^2 - i^2.BT^2 = OB^2 + BT^2 = OB^2 + BC^2 - CT^2 = OC^2 - CT^2 = OH.OL$ (C.2, Cor. 1) $= OH.MK$.

[†] Compare the example in § 266.

EXERCISES CI.

Solve the following equations :—

1. $x^2 + 5x - 7 = 0$. **2.** $x^2 - 7x - 3 = 0$. **3.** $x^2 + 7x + 5 = 0$.
4. $2x^2 + 5x - 2 = 0$. **5.** $2x^2 + 3x + 2 = 0$. **6.** $5x^2 - 8x - 10 = 0$.
7. $x^2 + x + 1 = 0$. **8.** $5x^2 - 12x + 15 = 0$.

Use Carlyle's construction to find :—

9. Two numbers whose sum is 20 and product 70.

10. Two numbers whose difference is 20 and product 70.

11. Two numbers whose sum is 5 and product 15.

12. Two numbers whose sum is 30 and product 300.

EXERCISES CII.

PRACTICAL GEOMETRY.

1. Draw a triangle whose sides measure 2, 3, 4, cms. Find geometrically the side of a square of equal area.

2. Construct a quadrilateral *ABCD* in which $AB = 2{\cdot}2''$, $\angle\ ABC = 80°$ $BC = 2''$, $\angle\ BCD = 20°$, $CD = 1{\cdot}6''$. Find the side of a square of equal area.

3. A rectangular field measures 330 yds. long and 220 yds. wide. Find by geometrical construction what would be the side of a square field of equal area.

4. Find by geometrical construction the height and width of a door whose area is 25 square feet, and whose perimeter is 21ft.

5. A line 10 miles long is divided internally in medial section. Find by geometrical construction the length of the smaller segment.

6. Without using the protractor find the area of an isosceles triangle each of whose equal sides measures 2ft. and whose vertical angle is $\frac{2}{5}$R.

RIDERS.

7. Construct a rectangle equal to a given rectangle and having one side three times the length of one side of the given rectangle.

8. Construct a rectangle of given perimeter equal to a given square.

9. On a given side construct a rectangle equal to a given square.

10. Of all rects. of given perimeter the square has the greatest area.

11. Of all rectangles of given area the square has the least perimeter.

12. Construct a rect. of given perimeter equal in area to a given rect.

13. Divide a given line so that the difference of the squares on the segments is equal to a given square. Hence solve graphically the simultaneous equations, $x^2 - y^2 = 7$, $x + y = 5$.

14. Divide a given line so that the sum of the squares on the segments is equal to a given square. Hence solve graphically the simultaneous equations, $x^2 + y^2 = 15$, $x + y = 5$.

15*. Divide a given line into two parts such that the square on one part is equal to three times the square on the other part.

CHAPTER XVIII.

REGULAR POLYGONS.

269. The names PENTAGON, HEXAGON, *heptagon*, OCTAGON, *nonagon*, *decagon*, *dodecagon*, *quindecagon*, are used for polygons contained by 5, 6, 7, 8, 9, 10, 12, and 15 sides respectively.

It is convenient to denote a polygon contained by n sides by the abbreviation "n-gon": similarly the abbreviation "7-gon" denotes a heptagon, and so on.

DEFINITION.—**A *regular* polygon is a polygon in which all the sides are equal and all the angles are equal.**

EXERCISES CIII.

1. Which of the following figures may be rightly described as regular?—

(i) An isosceles triangle, (ii) an equilateral triangle, (iii) a parallelogram, (iv) a rhombus, (v) a rectangle, (vi) a square.

2. Draw a figure to show that a pentagon may have all its sides equal but not all its angles equal.

Is such a pentagon regular?

3. If all the angles of a pentagon are equal, how many degrees are there in each? [Use Pn.1.]

Draw a pentagon having all its angles equal, but not all its sides equal. Is such a pentagon regular?

4. A pentagon is inscribed in a circle. If all its sides are equal, calculate the angle subtended by any one side at the centre. Hence calculate the magnitude of one angle of the pentagon.

Is the pentagon regular?

5. A polygon of n sides is inscribed in a circle. If all its sides are equal, calculate the angle subtended by any one side at the centre. Hence calculate the magnitude of one angle of the polygon.

Is the polygon regular?

6. An equilateral polygon is inscribed in a circle; prove that it is also equiangular. C.3; T.2.

7. Draw a circle of radius 1·5″. Using a protractor, inscribe a regular pentagon *ABCDE* in this circle. Measure *AB, BD.*

8. Draw a circle of radius 3 cms. Draw five radii, *OA, OB, OC, OD, OE,* at angular intervals of 72°. Draw the tangents at *A, B, C, D, E,* thus forming a pentagon circumscribed to the polygon. Take a trace of this figure on tracing paper.

Now apply the trace to the original figure in such a way that the trace of the point *O* falls on the point *O* itself, and the trace of the line *OA* falls on the line *OB.*

What do you notice, and what do you infer with regard to the circumscribed pentagon?

9. Draw a circle of radius 5 cms. Adjust the compasses by repeated trial till they "step out" the exact circumference in seven "steps." Hence inscribe a regular heptagon in the circle. Measure its side.

10. *ABCDEF* is a hexagon inscribed in a circle whose centre is *O.* Given that $\angle AOB = \angle COD = \angle EOF = 80°$, and $\angle BOC = \angle DOE = \angle FOA = 40°$; calculate the magnitude of each angle of the hexagon.

Is this polygon regular?

What contrast is suggested by the results of Questions 6 and 10?

270. THEOREM [Pn.2].—**If the circumference of a circle be divided into any number of equal parts, and if the points of division be joined in order, then the joining lines form a regular polygon inscribed in the circle.**

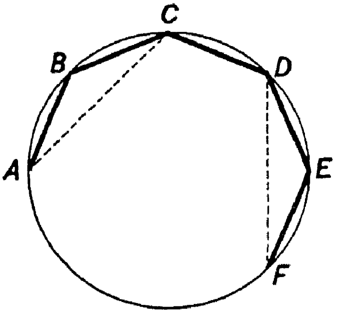

FIG. 404.

In Fig. 404 :—

Given that ○ᶜᵉ of the circle is divided into equal arcs by the points *A, B, C, D,* . . .

Required to prove that the polygon *ABCD* . . . is regular—*i.e.* that its sides are all equal and its angles are all equal.

Proof.—

(i) In ⊙ *ABC,* arcs *AB, BC, CD,* . . . are all equal;

hence chords *AB, BC, CD,* . . . are all equal. AC.6c.

(ii) Consider any two angles of the polygon, say $\angle B$ and $\angle E$. Join *AC, DF.*

$\because$ arcs *AB, BC, DE, EF* are all equal,

$\therefore$ arc *ABC* = arc *DEF,*

$\therefore$ chord *AC* = chord *DF.* AC.6c.

In $\triangle$s ABC, DEF, $\because \left\{\begin{array}{l} AB = DE, \\ BC = EF, \\ AC = DF, \end{array}\right.$

therefore $\angle B = \angle E$. C. 3.

Similarly we can prove that any two angles of the polygon $ABCD$ . . . are equal.

271. THEOREM [Pn.3].—**If the circumference of a circle be divided into any number of equal parts, and if tangents to the circle be drawn at the points of division, then these tangents form a regular polygon circumscribed about the circle.**

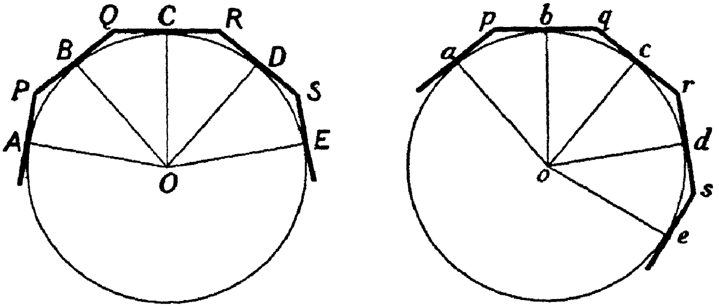

FIG. 405.

In Fig. 405 :—

Given that $\bigcirc^{ce}$ of the circle is divided into equal parts by the points $A, B, C,$ . . .

Required to prove that the polygon PQR . . . formed by the tangents at the points $A, B, C,$ . . . is regular—*i.e.* that its sides are all equal and its angles are all equal.

Cons.—Join $OA, OB, OC,$ . . .

Suppose that a duplicate of the figure is taken (*e.g.* by means of tracing paper). Let the points $o, a, b, c,$ . . . $p, q, r,$ . . . be the duplicates respectively of the points $O, A, B, C,$ . . . $P, Q, R,$ . . .

Now suppose that the duplicate is applied to the original figure in such a way that o falls on O, oa lies along OB, and p falls on the same side of OB as Q.

Proof.—Because, in the equal circles ABC, abc, arcs $AB, BC, CD, \ldots ab, bc, cd, \ldots$ are all equal,
Hyp. & Cons.

$\therefore \angle$s $AOB, BOC, COD, \ldots aob, boc, cod, \ldots$ are all equal. AC. 6*c*.

Thus, when oa lies along OB, ob lies along OC, oc along OD, and so on.

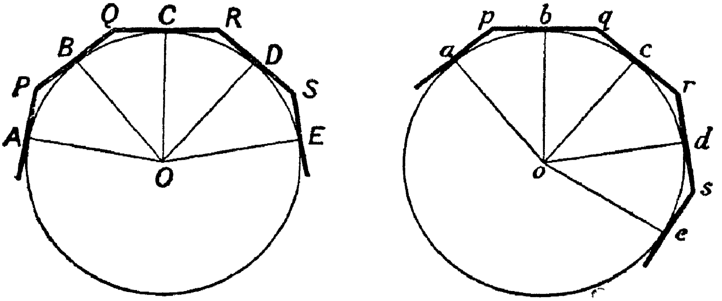

FIG. 405.

Again, $\because OA = OB = OC = \ldots = oa = ob = oc = \ldots$
$\therefore a$ falls on B, b falls on C, c falls on D, and so on.
$\because ap \perp oa$ and $PQ \perp OB$, $\therefore ap$ lies along PQ;
for otherwise we should have two different lines perpendicular to the same line OB at the same point B—which is impossible.

Similarly pq lies along QR, qr lies along RS, and so on.

Hence polygon $pqr \ldots$ coincides with polygon $QRS \ldots$

Thus $pq = QR$, $qr = RS$, ...,
i.e. $PQ = QR$, $QR = RS$, ...

Also $\angle p = \angle Q$, $\angle q = \angle R$, $\angle r = \angle S$, ...,
i.e. $\angle P = \angle Q$, $\angle Q = \angle R$, $\angle R = \angle S$, ...

Hence polygon $PQRS \ldots$ is regular.

272. **Alternative Proof of Theorem Pn.3.**—The following proof of Theorem Pn.3 is probably rather easier but is less instructive than the proof given in § 271.

THEOREM [Pn.3].—**If the circumference of a circle be divided into any number of equal parts, and if tangents to the circle be drawn at the points of division, then these tangents form a regular polygon circumscribed about the circle.**

In Fig. 406 :—*Given* that $\bigcirc^{ce}$ of the circle is divided into equal parts by the points $A, B, C, \ldots$.
Required to prove that the polygon $PQR \ldots$ is regular.

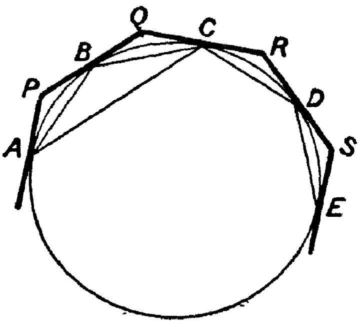

FIG. 406.

Cons.—Join $AC, AB, BC, CD, DE, \ldots$

Proof.—
$\because AP$ is a tangent and AB a chord, $\therefore \angle PAB = \angle BCA$. AC.5.
Similarly $\angle QBC = \angle BAC$. AC.5.
But $\angle BCA = \angle BAC$, for they stand on equal arcs AB, BC. AC.6*c*.
Hence $\angle PAB = \angle QBC$.
Similarly it can be proved that $\angle PBA = \angle QCB$.
Again, since arc AB = arc BC, $\therefore$ chord AB = chord BC. AC.6*c*.
Thus $\triangle PAB \equiv \triangle QBC$. C.2.
Similarly $\triangle QBC \equiv \triangle RCD \equiv \triangle SDE \equiv \ldots$
Hence $\angle P = \angle Q = \angle R = \angle S = \ldots$

It follows also from the congruence of the same triangles that

$$PA = QB = RC = SD = \ldots$$

and

$$PB = QC = RD = SE = \ldots$$

But the tangents PA, PB are equal ; Tn.2.

$$\therefore PA = PB = QB = QC = RC = RD = SD = SE = \ldots$$

Hence $PQ = QR = RS = \ldots$

273. THEOREM [Pn.4].—**The bisectors of the angles of a regular polygon all meet in a point: also with this point as centre circles can be circumscribed to, and inscribed in, the polygon.**

In Fig. 407 :—

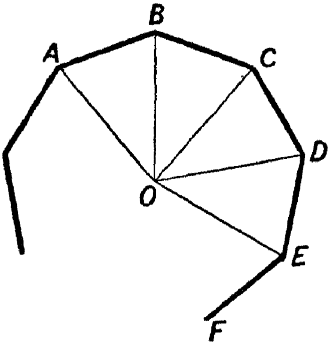

FIG. 407.

Given that $ABCD \ldots$ is a regular polygon—*i. e.* that all its sides are equal and all its angles are equal,

Required to prove that

(i) the bisectors of $\angle$s $A, B, C, \ldots$ all meet at some point O,

(ii) with O as centre a circle can be circumscribed to the polygon,

(iii) with O as centre a circle can be inscribed in the polygon.

(i) *Cons.*—Let the bisectors of $\angle$s A and B meet at O. Join $OC, OD, \ldots$

It will be sufficient to prove that $OC, OD, \ldots$ bisect $\angle$s $C, D, \ldots$ respectively.

Proof.—In $\triangle$s OBA, OBC,

$$\because \left\{\begin{array}{c} OB = OB \\ BA = BC \\ \angle OBA = \angle OBC \end{array}\right\}, \therefore \angle OCB = \angle OAB. \qquad \text{C.1.}$$

But $\angle OAB = \frac{1}{2}\angle A = \frac{1}{2}\angle C$; for $\angle A = \angle C$.

$$\therefore \angle OCB = \tfrac{1}{2}\angle C,$$

that is OC bisects $\angle C$.

Similarly it can now be shown that $\triangle OCD \equiv \triangle OCB$, and hence that OD bisects $\angle D$. And so on.

(ii) *Proof.*—In $\triangle OAB$, $\angle$s OBA, OAB are equal, being halves of equal angles.

Thus $OA = OB$. T.2c.

Similarly $OB = OC$, $OC = OD$, $OD = OE, \ldots$

Hence the circle described with centre O and radius OA will pass through the points $B, C, D, \ldots$ and will therefore be circumscribed to the polygon.

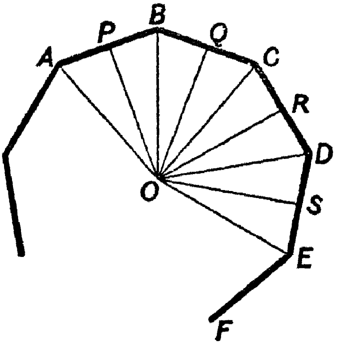

FIG. 408.

(iii) *Cons.*—From O draw $OP, OQ, OR, \ldots$ perpendicular to $AB, BC, CD, \ldots$ respectively (Fig. 408).

Proof.—In $\triangle$s OBP, OBQ, $\because \left\{ \begin{array}{c} OB = OB, \\ \angle OBP = \angle OBQ, \\ \angle OPB = \angle OQB, \end{array} \right\}$

$\therefore \quad OP = OQ.$ C.2.

Similarly $OQ = OR$, $OR = OS, \ldots$

Thus the circle described with centre O and radius OP will pass through the points $Q, R, S \ldots$

Moreover $AB, BC, CD, \ldots$ will be tangents to this circle, for they are perpendicular to the radii $OP, OQ, OR, \ldots$ respectively. Th.1

Hence this circle will be inscribed in the polygon.

COROLLARY.—**The lines which bisect the sides of a regular polygon at right angles all pass through one point, which is the centre of the inscribed and circumscribed circles.**

For it is easy to prove that the lines $OP, OQ, OR, \ldots$ in Fig. 408 bisect the sides of the regular polygon at right angles.

EXERCISES CIV.

Riders.

1. Show how to inscribe a square in a given circle.

2. Show how to circumscribe a square to a given circle.

3. Prove (without using Pn.4) that a circle can be circumscribed to a square, and that its centre will be the intersection of the diagonals of the square.

4. Prove (without using Pn.4) that a circle can be inscribed in a square, and that its centre will be the intersection of the diagonals of the square.

5. Show how to inscribe a regular octagon in a given circle, without using a protractor.

6. Show how to circumscribe a regular octagon about a given circle, without using a protractor.

7. Show how to inscribe in or circumscribe to a given circle regular polygons of 16, 32, 64, . . . sides, without using a protractor.

8. Prove that if a regular hexagon is inscribed in a circle, then each side is equal to a radius.

Hence show how to inscribe a regular hexagon in a given circle.

9. Prove Theorem Pn.2 by a proof similar to that of Theorem Pn.3 in § 271.

Problems in Practical Geometry.

10. Draw a circle of radius 4 cms. Inscribe a square and measure its side.

11. Draw a circle of radius 1·4″. Circumscribe a square and measure its diagonal.

12. Draw a circle of radius 1·6″. Inscribe a regular octagon. In the octagon inscribe a circle and measure its radius.

13. Draw a circle of radius 1·5″. Circumscribe a regular octagon. To the octagon circumscribe a circle and measure its radius.

14. Draw a circle of radius 1·5″. Using question 8, inscribe a regular hexagon. In the hexagon inscribe a circle and measure its diameter.

15. Draw a circle of radius 4 cms. Using question 8, circumscribe a regular hexagon. To the hexagon circumscribe a circle and measure its diameter.

16. AB is a side of a regular pentagon, and O is the centre of the circumscribed circle. Calculate the angles OAB, OBA.

Use this result to construct a regular pentagon $ABCDE$ on a given side measuring 1·4″. Measure the diagonal CE.

17. AB is a side of a regular 9-gon, and O is the centre of the circumscribed circle. Calculate the angles OAB, OBA.

Use this result to construct a regular 9-gon on a given side 1″ long. Measure its altitude.

274. PROBLEM.—**To inscribe a regular pentagon in a given circle.**

In Fig. 409 :—

Required to inscribe a regular pentagon in $\odot$ ABC.

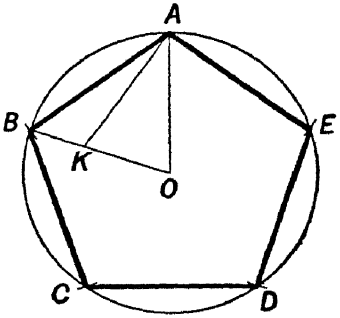

FIG. 409.

Cons.—Let O be the centre. Draw any radius OA.

With A as vertex and AO as side describe an isosceles triangle AOK, having each angle at the base double of the angle at the vertex. § 263.

Produce OK to meet $\bigcirc^{ce}$ at B. Join AB.

By adjusting the compasses to the length AB, set off chords BC, CD, DE, each equal to AB. Join AE.

Then $ABCDE$ shall be a regular pentagon.

Proof.—Join OC, OD, OE.

$\because$ $\angle AOB$ is one of the base angles of $\triangle AOK$;

$\therefore$ $\angle AOB = \frac{4}{5}R = \frac{1}{5}$ of 4R. § 263.

$\because$ chords AB, BC, CD, DE are all equal,

$\therefore$ $\angle$s AOB, BOC, COD, DOE, are all equal. Ch. 5.

Hence the sum of these four angles is $\frac{4}{5}$ of 4R ;

But the five angles at O together make up 4R ;

$\therefore$ remaining $\angle$ $EOA = \frac{1}{5}$ of 4R.

$\because$ $\angle$s AOB, BOC, COD, DOE, EOA are all equal,

$\therefore$ arcs AB, BC, CD, DE, EA are all equal. AC. 6.

Hence $ABCDE$ is a regular pentagon. Pn.2.

COROLLARY 1.—**To circumscribe a regular pentagon to a given circle.**

Use the same construction as before, and draw tangents at the points A, B, C, D, E.

These tangents will form a regular pentagon. Pn.3.

COROLLARY 2.—**To inscribe or circumscribe a regular polygon of 10, 20, 40, 80 . . . sides to a given circle.**

Bisect each of the arcs *AB, BC*, . . . in Fig. 409.

The $\bigcirc^{ce}$ is then divided into 10 equal parts, and hence a regular decagon can be inscribed or circumscribed to the ⊙. Pn.2, Pn.3.

Bisect each of the 10 equal arcs; $\bigcirc^{ce}$ is then divided into 20 equal parts. And so on.

275. **Alternative Methods.**—I. *Theoretically* the simplest method for inscribing a regular **decagon** in a circle (without using a protractor) is as follows (Fig. 410):—

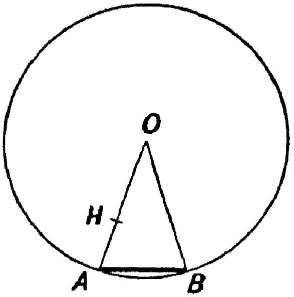

FIG. 410.

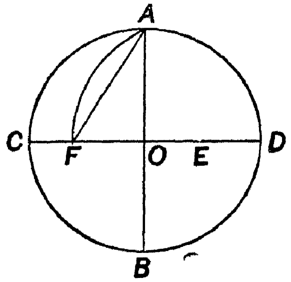

FIG. 411.

Divide the radius *OA* at the point *H* in medial section. Make the chord *AB* equal to *OH*, the larger segment of *OA*. Then, by § 263, *AB* is the side of the decagon.

For $\angle AOB = \frac{2}{5}R = \frac{1}{10} \times 4R$. Hence arc $AB = \frac{1}{10}\bigcirc^{ce}$.

Practically this is not a good method, because any small error in the size of the arc *AB* becomes multiplied by 9 as we step out the equal arcs round the circle, and the final discrepancy for the tenth arc may be serious.

II. The *best* method for inscribing a regular **pentagon** in a given circle is as follows (Fig. 411):—

Draw two perpendicular diameters *AOB, COD*. Bisect *OD* at *E*. With centre *E* and radius *EA* describe an arc meeting *OC* at *F*. Then *AF* is equal to the side of the required pentagon.

This method was not given in § 274 because it is very difficult to prove. It is easier to prove that F divides OC in medial section, and hence that OF is the side of the inscribed regular decagon.

276. **Accurate Methods for Inscribing or Circumscribing Regular Polygons to Circles.**—The problem of constructing a regular n-gon in or about a given circle resolves itself into the problem of dividing the $\bigcirc^{ce}$ into n equal arcs. Pn. 2 ; Pn. 3.

I. The $\bigcirc^{ce}$ can be divided into **4** equal arcs by drawing two perpendicular diameters, and thence into **8**, 16, 32, . . . equal arcs by repeated bisection.

II. The $\bigcirc^{ce}$ can be divided into **6** equal arcs by stepping out chords equal to the radius (compare Ques. 8, Exercises CIV); and thence into **12**, 24, 48, . . . equal arcs by repeated bisections. It can be divided into **3** equal arcs by taking the six equal arcs in pairs.

III. The $\bigcirc^{ce}$ can be divided into **5** equal arcs by the method of § 274 or § 275 and thence into **10**, 20, 30, . . . by repeated bisections.

IV. The $\bigcirc^{ce}$ can be divided into **15** equal arcs (and thence into 30, 60, . . .) as follows:—Construct an arc $AB = \frac{1}{3}\bigcirc^{ce}$ (Fig. 412), and an arc $AC = \frac{1}{5}\bigcirc^{ce}$. Bisect arc BC at D. Then arc $BC = \frac{1}{3} - \frac{1}{5}$ or $\frac{2}{15}\bigcirc^{ce}$. Hence arc $CD = \frac{1}{15}\bigcirc^{ce}$.

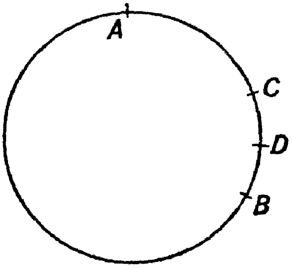

FIG. 412.

V. Gauss has proved that a purely geometrical construction is possible also for polygons of 17 and 257 sides, and generally for polygons of $2^n + 1$ sides provided n be a power of 2. The proofs are beyond the scope of this book. Two constructions for the 17-gon are given in the *Educational Times*, Mar. 1906.

277. Accurate Methods for Constructing a Regular Polygon given one side.—Methods have already been given for the *equilateral triangle* (§ 12), and the *square* (§ 116).

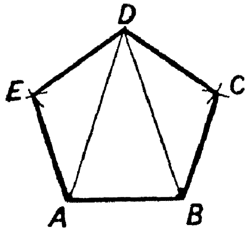

FIG. 413.

Pentagon.—To describe a regular pentagon on base AB (Fig. 413) :—

On AB as base describe an isosceles triangle DAB having each of the base angles double of the vertical angle (see § 264).

With A, D as centres and radius AB describe arcs intersecting at E: with B, D as centres and radius AB describe arcs intersecting at C. Then $ABCDE$ is the required regular pentagon.

[A strictly logical proof of this construction is difficult: it is, however, easy to show that if $ABCDE$ is a regular pentagon then DAB is an isosceles triangle having each of its base angles double of the vertical angle. A modification of this construction which is easily proved is given in Ques. 39, Exercises CVI.]

Hexagon.—To describe a regular hexagon on base AB:

On AB as base describe an equilateral triangle OAB. Then O is the centre of the circumscribed circle of the required hexagon. Draw this circle and step out chords equal to AB round its $\bigcirc^{ce}$. [Cf. Ques. 8, Exercises CIV.]

Other cases are given in the following Exercises :—

EXERCISES CV (Riders).

Prove the following constructions for Regular Polygons.

1. To inscribe an equilateral triangle in a given circle :—Through the centre O draw a diameter AD. With centre D and radius DO describe arcs cutting the circumference at B and C. ABC is the required equilateral triangle.

2. To inscribe a regular hexagon in a given circle :—Through the centre O draw a diameter AOD. With centres A and D and radius AO describe arcs cutting the circumference at points B, F, C, E. Then A, B, C, D, E, F are the angular points of the required hexagon.

3. To inscribe a regular 12-gon in a given circle :—Through the centre O draw two perpendicular diameters, AOB, COD. With centres A, B, C, D and radius AO describe arcs cutting the circumference in eight different points. These eight points together with A, B, C, D form the angular points of the required 12-gon.

4. To describe a regular octagon on a given base :—Bisect the given base AB at C: draw $CX \perp AB$. On CX mark off $CD = CA$: on DX mark off $DO = DA$. Then O is the circumcentre of the required octagon.

5. **To describe a regular decagon on a given base:**—On the given base AB describe an isosceles triangle OAB having each base angle double the angle at O. Then O is the circumcentre of the required decagon.

6*. To describe a regular octagon of given altitude:—Let AB be the given altitude. On base AB describe a square $ABCD$, whose diagonals intersect at K. With centres A, B, C, D and radius AK describe arcs cutting the sides of the square in eight different points. These points are the angular points of the required octagon.

278. **Approximate methods for constructing regular polygons in or about given circles:**—It is required to divide the circumference of a circle into n equal parts. There are three **general methods** available.

METHOD I. (*"Trial and Error" Method.*)

Adjust the compasses by repeated trials till they "step out" the exact circumference in n steps. See Ques. 9, Exercises CIII.

This method is capable of great accuracy if "screw" compasses are used, or better still "screw" dividers.

METHOD II.

Using the protractor draw n radii at angular intervals of $\frac{1}{n} \times 360°$.

METHOD III.

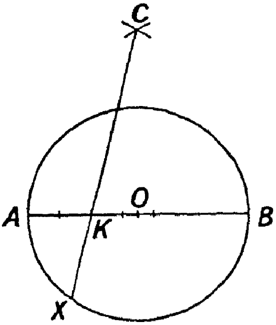

FIG. 414.

Draw a diameter AB (Fig. 414) and divide it into n equal parts (§ 115). Let K be the second division mark from A, and C the vertex of an equilateral triangle ABC. If CK produced meets $\bigcirc^{ce}$ in X, then

$$\text{arc } AX = \frac{1}{n} \bigcirc^{ce} \text{ (approximately).}†$$

It can be proved that this construction gives *accurate* results when n is 3, 4, or 6. The following table shows the percentage error in the arc AX for several other values of n (excess is indicated by the + sign, and defect by −):—

When n =	5,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16,	17.
Percentage error =	−·07,	+·2,	+·4,	+·7,	+1,	+1,	+2,	+2,	+2,	+2,	+3,	+3.

The error is 1 per cent., or under, up to and including the decagon.

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† This method was discovered by Renaldinus in the 17th century.

279. Approximate methods for constructing a regular polygon on a given side:—

METHOD I.

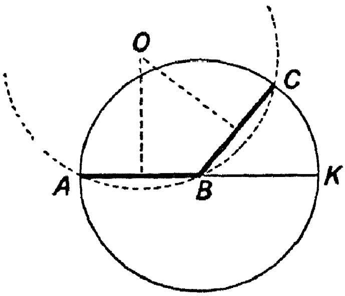

FIG. 415.

If AB is the given side (Fig. 415), draw a circle with centre B and radius BA, cutting AB produced in K. Mark off an arc $KC = \frac{1}{n}\bigcirc^{ce}$ (by the "trial and error" method or otherwise). Then BC is the adjacent side of the required polygon.

Bisect AB, BC at right angles by two lines meeting at O; then O is the centre of the circumscribed circle, and the required polygon is easily completed.

Explanation.

Exterior angle of the required regular polygon $= \frac{1}{n} \times 4\text{R}$. Pn. 1.

Since arc $CK = \frac{1}{n}\bigcirc^{ce}$, $\therefore \angle CBK = \frac{1}{n} \times 4\text{R}$.

Thus BC is in the correct position for the adjacent side of the polygon.

NOTE.—This method is *theoretically* exact if arc KC is exactly equal to $\frac{1}{n}\bigcirc^{ce}$. *Practically* it does not give good results.

METHOD II.

The method given in Ques. 16, 17 of Exercises CIV, can be used for any regular polygon.

280.* The following **special methods** are given by Mr. A. J. Pressland in Vol. LX of the *Educational Times*.

Regular 9-*gon*:—Draw a radius OA and bisect it at K. Draw a line KB meeting $\bigcirc^{ce}$ at B and making $\angle AKB = \frac{3}{4}\text{R}$. Then arc $AB = \frac{1}{9}\bigcirc^{ce}$. (Error, $-\cdot03\,\%$.)

Other regular polygons:—The side of the regular 11-gon inscribed in a circle is $\frac{1}{5}$ the diagonal of the circumscribed square. (Error, $+\cdot4\%$.)

If S_n represents the side of a regular n-gon inscribed in a circle, then

$$S_7 = \tfrac{1}{2} S_3 \text{ (Error, } +\tfrac{1}{5}\,\%\text{)};$$
$$S_{29} = \tfrac{1}{8} S_3 \text{ (Error, } +\tfrac{1}{8}\,\%\text{)}$$
$$S_{31} = \tfrac{1}{7} S_4 \text{ (Error, } -\tfrac{1}{6}\,\%\text{)}.$$

EXERCISES CVI.

PRACTICAL GEOMETRY.

Draw a circle of radius 3″: find the length of one side, and hence calculate the perimeter, of the following inscribed and circumscribed regular figures:—

1. Equilateral triangle. **2.** Square. **3.** Pentagon.
4. Hexagon. **5.**† Heptagon. **6.**† Octagon.
7.† 9-gon. **8.**† 10-gon. **9.**† 12-gon.
10.† 18-gon. **11.**† 30-gon.

12.† Which of these perimeters differs least from the circumference of the circle and by how much does it differ? [$\pi = 3{\cdot}14$.]

13. Draw a regular pentagon on a base of 2″ and find its area.

14. Draw a regular octagon on a base of 1·4″, and find its area.

15. Draw a regular 9-gon on a base 1″ and measure its altitude.

16. Draw an equilateral triangle of side 2″. Take a point A on one side of this triangle, ·5″ from one vertex. Inscribe a second equilateral triangle having one angular point at A. Measure its side.

17. Draw a square of side 1·5″. Inscribe an equilateral triangle in the square, one vertex of the triangle being at one corner of the square. Measure its side.

18. Inscribe a regular hexagon in an equilateral triangle of side 2″. Each side of the triangle is to contain two vertices of the hexagon. Measure the side of the hexagon.

19. Inscribe a regular pentagon in a circle of radius 2″. In this pentagon inscribe a regular decagon. Each side of the pentagon is to contain two vertices of the decagon. Measure the side of the decagon.

20. Draw an equilateral triangle ABC of side 2″. Draw a regular hexagon having A, B, C for alternate vertices, and measure its side.

21. Draw a square $ABCD$ of side 2″. Draw a regular octagon having A, B, C, D for alternate vertices, and measure its side.

CALCULATIONS.

22. Find the area of a square inscribed in a circle of radius r.

23. Find the area of a regular hexagon inscribed in a circle of radius r.

24.* Calculate the area of a regular octagon whose altitude is a.

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† An accurate result is difficult to obtain as a small error in the side may become serious when multiplied by the number of sides.

25. It is obvious that the perimeter and area of a regular 1000-gon circumscribed to a circle are practically equal to the circumference and area of the circle itself. Assuming this, find the area of a circle of radius r, given that the circumference is $2\pi r$. [See also Ques. 31.]

26. Find the area of an equilateral triangle of perimeter 24″.

27. Find the area of a square of perimeter 24″.

28. Find the area of a regular hexagon of perimeter 24″.

29.* Find the area of a regular octagon of perimeter 24″.

30. Find the area of a circle of perimeter 24″. [Use Ques. 25.] What fact is *suggested* by the results of Ques. 26-30 ?

RIDERS.

31. A polygon is circumscribed to a circle. Show that its area is equal to half the product of its perimeter and the radius.

32. *ABCDEFGH* is a regular octagon. Prove that *AB* is perpendicular to *CD.*

33. *ABCDEF* . . . is a regular polygon. Prove that *AD* is parallel to *BC.*

34. *ABCDEF* is a regular hexagon. Prove that *ABDE* is a rectangle.

35. *ABCDEFGH* is a regular octagon. Prove that *ABEF* is a rectangle.

36. An irregular polygon having an even number of sides is inscribed in a circle ; prove that the sum of one set of alternate angles is equal to the sum of the other set.

37. An irregular polygon having an even number of sides is circumscribed to a circle ; prove that the sum of one set of alternate sides is equal to the sum of the other set.

38. Prove the following construction for inscribing a regular pentagon in a given circle :—

Draw an isosceles triangle *XYZ,* having each of the base angles double of the vertical angle. In the given circle inscribe a triangle *ACD* equiangular to *XYZ.* Bisect the base angles *C, D* by lines meeting the circumference at *E, B.* Then *ABCDE* is the regular pentagon.

Also join *BE* and prove that every angle in the resulting figure has one of the values $\frac{2}{5}$R, $\frac{4}{5}$R, $\frac{6}{5}$R.

39. Prove the following construction for describing a regular pentagon on a given base *AB.*—On *AB* as base describe an isosceles triangle *DAB,* having each of the base angles double of the vertical angle. Draw the circle *ABD.* Bisect $\angle$s *A, B,* by lines meeting $\bigcirc^{ce}$ at *C, E.* Then *ABCDE* is the regular pentagon.

40. *ABCDE* . . . is a regular polygon, and *O* is the centre of the circumscribing circle. Prove that the circle through the points *O, B, C* touches the lines *AB, CD.*

41. An equiangular polygon is inscribed in a circle. Prove that alternate sides are equal. Hence prove that, if the number of sides is odd, all the sides are equal. AC.6, AC.6*c*.

42. *ABCDE* is a regular pentagon. A circle is described with centre *A* and radius *AC.* Prove that *CD* is the side of a regular decagon inscribed in this circle.

43. Any regular polygon is symmetrical about (i) the line which bisects any side at right angles, and (ii) the line which bisects any angle.

44. The line joining the middle point of one side of a regular pentagon to the opposite vertex is perpendicular to that side.

45.* The line joining the middle point of one side of a regular 11-gon to the opposite vertex is perpendicular to that side.

46.* Show how to describe a pentagon given one diagonal.

47.* Draw a circle and divide its $\bigcirc^{ce}$ into five equal parts at the points *A, B, C, D, E.* Join *AC, CE, EB, BD, DA.* These straight lines form a star-shaped figure,† consisting of a central pentagon surrounded by five triangles. Prove that this pentagon is regular. [Rotate the figure about the centre of the circle.]

EXERCISES CVII.

Miscellaneous Riders.

Rectangle-Theorems.

1. *ABC* is an equilateral triangle and *D* any point in *BC;* prove that the square on *BC* is equal to the rectangle (*BD, DC*) together with the square on *AD.*

2. State and prove the theorem corresponding to Ques. 1 when *D* is in *BC* produced.

3. If *DE* be drawn parallel to the base *BC* of an isosceles triangle *ABC,* the square on *BE* is equal to the square on *CE,* together with the rectangle contained by *BC, DE.*

4. *AB* is a given diameter of a circle, and *CD* any parallel chord; if any point *E* be taken in *AB,* and *E* be joined to *C* and *D,* prove that the sum of the squares on *EA, EB* is equal to the sum of the squares on *EC, ED.*

5. If the tangents from a given point to any number of intersecting circles are equal, all the common chords of the circles pass through that point.

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† Known as the pentagram and supposed in ancient times to have magic properties.

6. Two circles whose centres are at A and B intersect at right angles, and their common chord meets AB in O; if CD is any chord of the first circle passing through B, show that the four points A, C, D, O are concyclic.

7. Divide a given straight line into two parts, so that the rectangle contained by the parts may be a maximum.

8. Divide a given straight line into two parts so that the sum of the squares on the two parts may be a minimum.

9. Divide a given straight line into three parts, so that the sum of the squares on the three parts may be a minimum.

10. From a point P outside a circle draw a line PQR meeting the circle and such that $PR.QR$ is equal to a given square.

11.* AB is a diameter of a circle, CD a chord $\perp$ AB; a line APQ meets the circle in Q and CD in P. Prove that $AP.AQ$ is constant.

12.* A semicircle is described on AB. Any two chords AD, BC intersect in P; prove $AB^2 = AD.AP + BC.BP$.

13.* If the common tangents of two intersecting circles meet the common chord produced in X and Y, show that the sum of the squares on a common tangent and on the common chord is equal to the square on XY.

14.* In any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the straight lines joining the middle points of the opposite sides.

15.* In any quadrilateral, two of whose opposite sides are parallel, the sum of the squares on the diagonals is equal to the sum of the squares on its non-parallel sides, together with twice the rectangle contained by the parallel sides.

16.* ABC is any triangle and its medians intersect in G; if P be any point, show that the sum of the squares on AP, BP, CP is greater than the sum of the squares on AG, BG and CG by three times the square on GP.

17.* ABC is any triangle; find a point P within it such that the sum of the squares on AP, BP and CP is a minimum.

18.* Find the position of a point such that the sum of the squares on its distances from four given fixed points is a minimum.

19*. If A, B, C, D are fixed points, and P is a point such that the sum of the squares on PA, PB is equal to the sum of the squares on PC, PD, prove that the locus of P is a straight line at right angles to the line joining the middle points of AB, CD.

20.* In the last question prove that the locus of P passes through the intersection of the lines drawn perpendicular to either of the pairs of lines AC, BD or AD, BC at the middle points.

21.* $ABCD$ is a $\|^m$. A circle through A cuts AB, AC, AD in F, G, H. Prove that $AB.AF + AD.AH = AC.AG$.

On Polygons.

22. What are the values of n (<20) for which it is impossible to describe a regular polygon of n sides by a purely geometrical construction?

23. Prove that the area of the regular hexagon $ABCDEF$ is double the area of an equilateral triangle inscribed in the same circle.

24. From the vertices of a given triangle draw straight lines which shall form an equilateral hexagon, whose area shall be double that of the triangle.

25. Show that the straight lines, which join the alternate vertices of a regular hexagon, intersect to form another regular hexagon, the area of which is one-third the area of the original hexagon.

26. Show that four times the area of a regular hexagon inscribed in a circle is equal to three times the area of the corresponding circumscribed hexagon.

27. If two equilateral triangles have a common inscribed circle, prove that the points of intersection of the sides form an equilateral hexagon.

28. If the alternate angles of an equilateral hexagon be equal to one another, a circle can be inscribed in the hexagon.

29.* $ABCDEF$ is a regular hexagon inscribed in a circle; another circle is described passing through A and B, and cutting the first circle orthogonally; and a third circle is described through A and C, and cutting the first circle orthogonally; show that the radius of the second circle is one-third of the radius of the third circle.

30.* If any point P be taken within the regular n-gon $ABCDE\ldots$, and perpendiculars be drawn from P to each of the sides, prove that the sum of these perpendiculars is equal to n times the radius of the inscribed circle.

31.* $ABCDE$ is a regular pentagon, and AC and BE meet in F; show that the circle AOF touches the side AB, where O is the centre of the circum-circle of the pentagon.

32.* $ABCDE$ is a regular pentagon, and AC, BD intersect at K; prove that $AK=AB$, and hence that K divides AC in medial section.

33.* Show that the difference of the squares on a diagonal and on a side of a regular pentagon is equal to the rectangle contained by them.

34.* Using Fig. 409, prove that the square on the side of a regular pentagon inscribed in a circle is the sum of the squares on the sides of the regular inscribed hexagon and decagon. Hence prove the construction for the pentagon given in § 275.

LIST OF PROPOSITIONS.

THE chief object of this list of Propositions is to serve for an index to the reference symbols used throughout the book. As the Propositions are given in their logical order, the list will also be found useful as a summary of the course of Theoretical Geometry.

In order that this summary should be complete, the contents of Chapters V, VII, XIII, and XVII (Problems) and of Chapter XIV (Miscellaneous Propositions) have been included. It should however be remembered that none of these form an essential part of the logical course, and that the contents of Chapter XIV cannot claim to rank as Propositions in the usual acceptation of the term.

CHAPTER II.—ANGLES AND PARALLELS.

Angles. PAGE

A.1.—The two angles which one straight line makes with another straight line on one side of it are together equal to two right angles . . . . (*Euc. I. 13*) 83

A.1*c*.—If at a point in a straight line, two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in the same straight line . . . . (*Euc. I. 14*) 84

A.2.—Each of the angles formed by two intersecting straight lines is equal to the vertically opposite angle . (*Euc. I. 15*) 86

Parallels.

P.1.—If a straight line meeting two other straight lines in the same plane makes two alternate angles equal to one another, these two straight lines shall be parallel . . . (*Euc. I. 27*) 88

P.2.—If a straight line, meeting two other straight lines, makes an exterior angle equal to the interior and opposite angle on the same side of the line; or if it makes the two interior angles on the same side together equal to two right angles; then the two straight lines shall be parallel . . (*Euc. I. 28*) 90

PLAYFAIR'S AXIOM.—It is not possible that there should be two straight lines drawn through the same point and parallel to the same line 91

P.1*c*.—If a straight line meets two parallel straight lines, it shall make the alternate angles equal to one another (*Euc. I. 29*) 91

P.2*c*.—If a straight line meets two parallel straight lines, it shall make each exterior angle equal to the interior and opposite angle on the same side of the line, and each pair of interior angles on the same side of the line together equal to two right angles (*Euc. I. 29*) 92

PAGE

P.3.—If two straight lines are parallel to the same straight line they are parallel to one another . . . (*Euc. I. 30*) 93

P.4.—If the two arms of one angle are respectively parallel to the two arms of another, then the angles are either equal or supplementary 94

P.5.—[See Chapter IV.]

CHAPTER III.—TRIANGLES.

Triangles.

T.1.—If any side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior and opposite angles: also, the three angles of a triangle are together equal to two right angles . . . . (*Euc. I. 32*) 105

RT.1.†—In a right-angled triangle the two acute angles are together equal to one right angle. 106

Polygons.

Pn.1.—If all the sides of a convex polygon be produced in the same direction round the polygon, the exterior angles so formed are together equal to four right angles; and in any convex polygon the sum of the interior angles, together with four right angles, is equal to twice as many right angles as the figure has sides . . . . (*Euc. I. 32, Cor.*) 110

Congruence

C.1.—If two triangles have two sides and the included angle in the one respectively equal to two sides and the included angle in the other, then shall these triangles be congruent. (*Euc. I. 4*) 114

C.2.—If two triangles have two angles and one side in the one triangle respectively equal to two angles and the corresponding side in the other triangle, then the two triangles shall be congruent . . . . (*Euc. I. 26*) 116

Triangles.

T.2.—If two sides of a triangle are equal, then shall the angles opposite to these sides be equal . . . . (*Euc. I. 5*) 118

T.2*c*.—If two angles of a triangle are equal, then the sides opposite to these angles are also equal . . (*Euc. I. 6*) 119

Congruence.

C.3.—If the three sides of one triangle are respectively equal to the three sides of another triangle, the two triangles shall be congruent . . . . (*Euc. I. 8*) 120

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† For the remaining Theorems on Right-angled Triangles see Chapter IX.

PAGE

C.4.—If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles shall be congruent 122

C.5.—If two sides of one triangle are respectively equal to two sides of another, and if the angles opposite to one pair of equal sides are equal, then shall the angles opposite to the other pair of equal sides be either equal or supplementary.† (Cf. *Euc. VI. 7*) 125

Triangles.

T.3.—If one side of a triangle is greater than another, then the angle opposite to the first side shall be greater than the angle opposite to the second . . . . . . (*Euc. I. 18*) 123

T.3*c*.—If one angle of a triangle is greater than another, then the side opposite to the first angle shall be greater than the side opposite to the second . . . . . . (*Euc. I. 19*) 123

T.4.—Any two sides of a triangle are together greater than the third . . . . (*Euc. I. 20*) 124

T.5.—The shortest line which can be drawn from a given point to a given line is the line which is perpendicular to the given line; and of any other two lines drawn from the given point to the given line on one side of the perpendicular, that which is nearer to the perpendicular is less than that which is more remote . . 128

T.6.—If the two sides of one triangle are respectively equal to the two sides of another, and if the vertical angle of the first triangle is greater than the vertical angle of the second triangle; then the base of the first triangle shall be greater than the base of the second . . . . (*Euc. I. 24*) 129

T.6*c*.—If the two sides of one triangle are respectively equal to the two sides of another, and if the base of the first triangle is greater than the base of the second triangle; then the vertical angle of the first triangle shall be greater than the vertical angle of the second . . . . (*Euc. I. 25*) 130

T.7.—[See Chapter IV.]

CHAPTER IV.—PARALLELOGRAMS.

Parallelograms.

Pm.1.—In any parallelogram, opposite sides are equal, and opposite angles are equal; also each diagonal divides the parallelogram into two congruent triangles . . . . (*Euc. I. 34*) 136

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† The *Corollary* to this proposition depends on T.3.

PAGE

Pm.1c.—(i) If the opposite sides of a quadrilateral are equal, it is a parallelogram; also, (ii) if the opposite angles of a quadrilateral are equal it is a parallelogram 137

Pm.2.—If in a quadrilateral two opposite sides are equal and parallel, then the quadrilateral is a parallelogram, and the other two sides are also equal and parallel . . (*Euc. I. 33*) 138

Pm.3.—If one angle of a parallelogram is a right angle so are all the others 138

Pm.4.—The diagonals of a parallelogram bisect each other . 139

Pm.5.—The diagonals of a rhombus or square bisect its angles and are perpendicular to each other 139

Pm.6.—The diagonals of a rectangle or square are equal . . 140

Pm.7.—If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and have also the angles included between those sides equal, then the two parallelograms shall be congruent 141

Pm.8.—[See page 196, Ques. 13.]

Parallels.†

P.5.—If there are three or more parallel straight lines and if the intercepts made by them on any one transversal are equal, then the intercepts made by them on any other transversal are equal 142

Triangles.‡

T.7.—(i) The line which joins the middle points of the sides of a triangle is parallel to, and half the length of, the base; and (ii) the line drawn parallel to the base, from the middle point of one side, bisects the other side and is half the length of the base (Cf. *Euc. VI. 2*) 143

CHAPTER V.—PROBLEMS.

To bisect a given angle. . . . . . (*Euc. I. 9*) 147

At a given point in a given straight line to construct an angle equal to a given angle . . . . . . (*Euc. I. 23*) 149

To draw a line through a given point parallel to a given straight line . . . . (*Euc. I. 31*) 150

To draw a line perpendicular to a given straight line from a given point within it . . . . . . (*Euc. I. 11*) 151

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† For the remaining Theorems on Parallels see Chapter II.

‡ For the remaining Theorems on Triangles see Chapters III, XVI.

PAGE

To bisect a given straight line . . . . (*Euc. I. 10*) 152

To draw a line perpendicular to a given straight line from a given point without the line . . . . . (*Euc. I. 12*) 153

To divide a given straight line into five equal parts (Cf. *Euc. VI. 9, 10*) 154

To construct a square given one side . . (*Euc. I. 46*) 155

CHAPTER VI.—LOCI.

L.1.—The locus of a point which is at a given distance from a given point is (the circumference of) a circle whose centre is the given point and whose radius is the given distance . . 163

L.2.—The locus of a point which is at a given distance from a given line, on one side of it, is a line drawn parallel to the given line through any point which lies at the given distance from the given line 164

L.3.—The locus of a point which is equidistant from two given points is the straight line which bisects at right angles the line joining the two given points 166

L.4.—The locus of a point which is equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the given lines 166

L.5.—[See Chapter VII.] **L.6.**—[See Chapter XII.]

L.7.—[See Chapter XVI.] **L.8.**—[See Chapter XVI.]

CHAPTER VII.—AREAS.

Areas.

Ar.1.—The area of a rectangle is given by the product of the lengths of two adjacent sides 178

Ar.2.—Parallelograms on the same base and between the same parallels are equal in area . . . . . . (*Euc. I. 35*) 184

Ar.3.—If a parallelogram and a triangle be on the same base and between the same parallels then the area of the parallelogram shall be double that of the triangle . . . . (*Euc. I. 41*) 186

Ar.4.—Parallelograms on equal bases and between the same parallels are equal in area . . . . (*Euc. I. 36*) 187

Ar.5.—Triangles on equal bases and between the same parallels are equal in area . . . . (*Euc. I. 38*) 188

Ar.5*c*.—Equal triangles on equal bases in the same straight line, and on the same side of it, are between the same parallels (*Euc. I. 40*) 188

PAGE

Loci.†

L. 5.—The locus of the vertex of a triangle of given area and standing upon a given base consists of two straight lines parallel to the base 190

CHAPTER VIII.—PROBLEMS ON AREA.

To divide a parallelogram or triangle into any given number of equal parts 199

To construct a triangle equal to a given convex polygon . . 200

To construct a triangle equal to a given triangle, given one side and an adjacent angle of the required triangle . . . 202

CHAPTER IX—THE RIGHT-ANGLED TRIANGLE.

RT. 1.—See Chapter III.

RT. 2.—In a right-angled triangle the square described on the hypotenuse is equal in area to the sum of the squares described on the other two sides . . . . (*Euc I. 47*) 212

RT. 2c.—If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, the angle contained by these two sides is a right angle. (*Euc. I. 48*) 213

RT. 3.—In a right-angled triangle the square on either side containing the right angle is equal to the rectangle contained by the hypotenuse and the projection of that side on the hypotenuse (*Euc. VI. 8 Cor.*) 213

CHAPTER X.—THE CIRCLE: SYMMETRY.

Sy. 1.—If two points are symmetrical about a given axis, then (i) their join is bisected at right angles by this axis, and (ii) their joins to any point on the axis are equal, and form an angle which is bisected by the axis 222

Sy. 1c.—Two points are symmetrical to a given axis if (i) their join is bisected at right angles by this axis, or (ii) their joins to some point on the axis are equal, and form an angle which is bisected by that axis 223

Sy. 2.—A circle is symmetrical about any diameter . . 223

Sy. 3.—If a pair of points is symmetrical to a second pair, then the joins of these pairs either are parallel to, or meet on, the axis of symmetry 224

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† For the remaining theorems on Loci see Chapters VI, XII, XVI.

CHAPTER XI.—THE CIRCLE: CHORDS AND TANGENTS.

Chords.

PAGE

Ch.1.—If a diameter of a circle bisects a chord which is not a diameter, it is perpendicular to that chord; and conversely if the diameter is perpendicular to the chord it bisects it (*Euc. III. 3*) 232

Ch.2.—The line which bisects any chord of a circle at right angles passes through the centre . . . . (*Euc. III. 1*) 233

Ch.3.—It is possible to draw one, and only one, circle through three points which do not lie in a straight line (Cf. *Euc. IV. 5*) 234

Ch.4.—In any circle equal chords are equidistant from the centre; and conversely, chords which are equidistant from the centre are equal . . . . . . (*Euc. III. 14*) 236

Ch.5.—In any circle, if two chords are equal they subtend equal angles at the centre; and conversely chords which subtend equal angles at the centre are equal . . (Cf. *Euc. III. 28*) 237

Ch.6.—In any circle the diameter is the greatest chord. And of any other chords, a chord which is nearer to the centre is greater than one which is more remote; and conversely, a chord which is greater than another is nearer to the centre . (*Euc. III. 15*) 238

Ch.7.—If two circles cut one another, the line joining their centres bisects the common chord at right angles † . . . 243

Tangents.

Tn.1.—The line drawn through any point on the circumference of a circle at right angles to the radius through that point is a tangent to the circle; and any other line drawn through that point is a secant . . . . (*Euc. III. 16*) 240

Tn.2.—If two tangents are drawn to a circle from the same point outside the circle, they are equal in length; also the line joining this point to the centre bisects the angle between the tangents and the angle between the radii to the points of contact of the tangents, and bisects at right angles the line joining the two points of contact 241

Tn.3.—If two circles touch one another then (i) the join of their centres passes through the point of contact, and (ii) the distance between their centres is equal to the sum or difference of their radii . . . . . . (*Euc. III. 11, 12*) 244

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† In the text Ch 7 has been placed immediately before Tn. 3, to which it is closely related.

CHAPTER XII.—The Circle: Angular Properties.

Angular Properties. PAGE

AC.1.—The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining arc of the circumference . . . (*Euc. III. 20*) 254

AC.2.—Angles in the same segment of a circle are equal (*Euc. III. 21*) 255

AC.2*c*.—If the line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic 256

Loci.†

L.6.—The locus of the vertex of a triangle drawn on one side of a given base and having a given vertical angle is the arc of a segment of a circle on that base 257

Angular Properties.

AC.3.—If a quadrilateral be inscribed in a circle the sum of each pair of opposite angles is equal to two right angles (*Euc. III. 22*) 257

AC.3*c*.—If two opposite angles of a quadrilateral are together equal to two right angles then the vertices of the quadrilateral are concyclic 258

AC.4.—(i) Any angle in a semicircle is a right angle. (ii) Any angle in a segment greater than a semicircle is less than a right angle. (iii) Any angle in a segment less than a semicircle is greater than a right angle . . . . . (*Euc. III. 31*) 260

AC 5.—If a chord be drawn from the point of contact of a tangent to a circle, then each of the angles which the chord makes with the tangent is equal to the angle in the alternate segment of the circle . . . . (*Euc. III. 32*) 261

AC.5*c*. If a line drawn through one extremity of a chord of a circle makes an angle on one side of the chord equal to the angle in the alternate segment, then this line is a tangent to the circle 262

AC.6.—In equal circles if two arcs subtend equal angles at the centre they are equal . . . . . . (*Euc. III. 26*) 262

AC.6*c*.—In equal circles, if two arcs are equal they subtend equal angles at the centres . . . . . (*Euc. III. 27*) 264

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† For the remaining theorems on Loci see Chapters VI, VII, XVI.

CHAPTER XIII.—THE CIRCLE: PROBLEMS.

PAGE

To bisect a given arc . . . . . . (*Euc. III. 30*) 277

Given an arc of a circle, to find its centre . (*Euc. III. 25*) 277

From a given point to draw a tangent to a given circle (*Euc. III. 17*) 278

To draw the direct common tangents to two unequal circles 280

To draw the transverse common tangents to two given circles 281

To circumscribe a circle to a given triangle . (*Euc. IV. 5*) 282

To draw a circle touching three given lines which are neither concurrent nor all parallel . . . . . . (*Euc. IV. 4*) 282

From a given circle to cut off a segment to contain a given angle . . . . (*Euc. III. 34*) 285

On a given base to draw a segment of a circle to contain a given angle . . . . (*Euc. III. 33*) 286

In a given circle to inscribe a triangle equiangular to a given triangle . . . . (*Euc. IV. 2*) 288

To circumscribe to a given circle a triangle equiangular to a given triangle . . . . (*Euc. IV. 3*) 288

CHAPTER XIV.—MISCELLANEOUS PROPOSITIONS.

The three lines which bisect the sides of a triangle at right angles are concurrent 304

The three lines which bisect the angles of a triangle are concurrent 305

In any triangle the three medians are concurrent 306

The three lines drawn from the vertices of a triangle perpendicular to the opposite sides are concurrent 309

ABC is a triangle and P is any point upon its circum-circle. PD, PE, PF are perpendicular to the sides BC, CA, AB respectively; prove that D, E, F are collinear 311

If two circles cut one another they form the same angle at each intersection. 312

Of all triangles of given area upon a given base the isosceles triangle is the one of minimum perimeter 315

Of all triangles of given area the equilateral triangle has the minimum perimeter 315

Of all quadrilaterals of given area the square has the minimum perimeter 316

CHAPTER XV.—THE COMPOSITION OF RECTANGLES.

PAGE

R.1.—If a rectangle be divided (internally or externally) by lines parallel to its sides then any of the quadrilaterals so formed is a rectangle, and its opposite sides are equal 331

R.2.—The rectangle contained by any given line and the sum (or difference) of two other given lines is equal to the sum (or difference) of the rectangles contained by the first line and each of the other two lines . . . . . . (*Euc. II. 1*) 334

R.3.—If a line be divided *internally* at any point, then the square on the line is equal to the sum of the squares on the segments *plus* twice the rectangle contained by the segments (*Euc. II. 4*) 335

R.4.—If a line be divided *externally* at any point, then the square on the line is equal to the sum of the squares on the segments *minus* twice the rectangle contained by the segments (*Euc. II. 7*) 336

R.5.—The difference between the squares on two given lines is equal to the rectangle contained by the sum and the difference of the two given lines . . . . (*Euc. II. 5, 6*) 338

R.6.—The sum of the squares on the sum and difference of two lines is equal to twice the sum of the squares on the two lines; also the difference of the squares on the sum and difference of two lines is equal to four times the rectangle contained by the two lines . . . . (*Euc. II. 8, 9, 10*) 339

CHAPTER XVI.—RECTANGLE-THEOREMS ON TRIANGLES AND CIRCLES.

On Triangles.†

T.8.—In an *obtuse-angled* triangle the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle *plus* twice the rectangle contained by one of these two sides and the projection of the other upon it (*Euc. II. 12*) 348

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† For the remaining Theorems on Triangles see Chapters III, IV.

PAGE

T.9.—In *any* triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the two sides containing that angle *minus* twice the rectangle contained by one of these two sides and the projection of the other upon it (*Euc. II. 13*) 349

T.10.—The sum of the squares on the two sides of a triangle is equal to twice the sum of the squares on the half of the base and the median which bisects the base. Also, the difference between the squares on the sides of any triangle is equal to twice the rectangle contained by the base and the projection on the base of the median which bisects it 354

Loci.†

L.7.—The locus of a point, which moves so that the sum of the squares on its distances from two fixed points is constant, is a circle 354

L.8.—If a point moves so that the square on its distance from one fixed point exceeds the square on its distance from another by a constant area, then the locus of the moving point is a line perpendicular to the join of the fixed points 355

On Circles.

RC.1.—If two chords of a circle intersect at a point within the circumference, then the rectangle contained by the segments of the one chord shall be equal to the rectangle contained by the segments of the other . . . . . . (*Euc. III. 35*) 358

RC.1*c*.—If two lines divide each other internally so that the rectangle contained by the segments of the one line is equal to that contained by the segments of the other, then the four extremities of the two lines are concyclic points 359

RC.2—If two chords of a circle intersect at a point without the circumference, then the rectangle contained by the segments of the one chord shall be equal to the rectangle contained by the segments of the other 360

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† For the remaining theorems on Loci see Chapters VI, VII, XII.

PAGE

RC.3.—If a chord of a circle is divided externally by a tangent, the rectangle contained by the segments of the chord is equal to the square on the tangent . . . . . (*Euc. III. 36*) 362

RC.2*c*.—If two lines divide each other externally so that the rectangle contained by the segments of the one line is equal to that contained by the segments of the other, the four extremities of the two lines are concyclic points 362

RC.3*c*.—If a chord of a circle is divided externally at a given point and if a line is drawn from this point to the circumference such that the square on this line is equal to the rectangle contained by the segments of the chord, then this line touches the circle (*Euc. III. 37*) 363

Radical Axis.

The radical axis of two circles is a straight line perpendicular to the join of their centres 366

To construct the radical axis of two given circles . . . 367

CHAPTER XVII.—Problems on Rectangles.

To construct a square equal to a given rectangle (*Euc. II. 14*) 369

To divide a given line internally so that the rectangle contained by the segments is equal to the square on a given line . . 369

To divide a given line externally so that the rectangle contained by the segments is equal to the square on a given line . . 370

To divide a given line internally in medial section (*Euc. II. 11*) 372

To divide a given line externally in medial section . . . 372

To construct an isosceles triangle having each base angle double of the vertical angle, given one side of the triangle (*Euc. IV. 10*) 374

To construct an isosceles triangle having each base angle double of the vertical angle, given the length of the base . . . 375

CHAPTER XVIII.—REGULAR POLYGONS.

PAGE

Pn.2.†—If the circumference of a circle be divided into any number of equal parts, and if the points of division be joined in order, then the joining lines form a regular polygon inscribed in the circle . . . . (Cf. *Euc. IV. 11*) 382

Pn.3.—If the circumference of a circle be divided into any number of equal parts, and if tangents to the circle be drawn at the points of division, then these tangents form a regular polygon circumscribed about the circle . . . . (Cf. *Euc. IV. 12*) 383

Pn.4.—The bisectors of the angles of a regular polygon all meet in a point: also with this point as centre circles can be circumscribed to, and inscribed in, the polygon (Cf. *Euc. IV. 13, 14*) 386

To inscribe a regular pentagon in a given circle (*Euc. IV. 11*) 389

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† For Theorem Pn.1, see Chapter III.

ANSWERS.

EXERCISES I (*Page* 5).

9. B. **10.** E. **16.** (i) No. (ii) At Q.

EXERCISES II (*Pages* 7, 8).

1. 4·2″, 1·9″, 4·8″. **2.** 9·4 cms., 6·6 cms., 13·4 cms. **3.** 3·7″, 2·6″, 5·3″. **4.** 10·7 cms., 4·8 cms., 12·2 cms. **5.** ·7″, 1·3″, 2·6″, 3·2″. **6.** 1·4 cms., 2·8 cms., 4·1 cms., 5·7 cms., 7·5 cms.

EXERCISES III (*Pages* 12, 13).

1. 2·53″, 6·43 cms. **2.** 3·47″, 8·81 cms. **3.** 2·75″, 6·99 cms.
4. 1·87″, 4·74 cms. **5.** 0·56″, 1·42 cms. **6.** 2·22″, 5·64 cms.
7. 2·58″, 6·55 cms. **8.** 2·31″, 5·86 cms. **9.** 1·88″, 4·77 cms.
10. 1·10″, 2·80 cms. **11.** 3·27″, 8·31 cms. **12.** 0·82″, 2·09 cms.
13. 23·8 miles. **14.** 6·4 miles. **15.** 19·4 miles.
16. 15·2 miles.

EXERCISES IV (*Page* 16).

11. Impossible. **12.** Impossible.

EXERCISES V (*Page* 19).

1. ACB (or BCA); DEF (or FED); HGK (or KGH); LNM (or MNL).
2. ACB (or BCA); DFE (or EFD); GHK (or KHG); LNM (or MNL).
3. PQ and PR. **4.** YX and YZ. **5.** PQR (or RQP).
6. ZYX (or XYZ). **7.** PQR (or RQP). **8.** QPR (or RPQ)
9. XZY (or YZX). **10.** Greater. **12.** E is the greatest and B the smallest. **13.** D is the greatest and M the smallest.
14. Angle H = angle K. **15.** All three angles are equal.

EXERCISES VI (*Page* 20).

1. (*a*) Twice, (*b*) four times, (*c*) six times, (*d*) ten times. **2.** PG. **3.** PF. **4.** PF. **5.** Twice. **6.** Twice. **7.** Five times. **8.** BPD (or DPB), DPG (or GPD). **9.** BPD (or DPB). **10.** BPG (or GPB).

EXERCISES VII (*Pages* 26, 27).

1. 1·225″. **2.** ·75″, 1·25″. **3.** 1·79″. **4.** 23·6 mms. **5.** 46·8 mms. **6.** 1·73″, ·67″. **7.** 19·7 mms. **8.** 4 cms., 4 cms. **9.** 1″; 1·8″, 1·8″. **10.** 1·65″.

EXERCISES VIII (*Pages* 32, 33).

1. 75·5°, 29°, 75·5°. **2.** 53·1°, 90°, 36·9°. **3.** 117·3°, 26·4°, 36·3°.
4. 21·8°, 38·2°, 120°. **5.** 60°, 60°, 60°. **6.** 90°, 67·4°, 22·6°.
7. 25·6°, 120 2°, 34·2°. **8.** 106 2°, 36·9°, 36·9°.
9. 56·3°, 33·7°, 90°. **10.** 63·5°, 36·7°, 79·8°.

11. The three angles of any triangle together make up 180°.

12. In any triangle the greatest angle is opposite to the greatest side, and the least angle to the least side.

13. If two sides of a triangle are equal, the angles opposite to these sides are also equal.

14. If the three sides of a triangle are equal, then the three angles are also equal.

15. Of the four angles formed by two intersecting lines, each angle is equal to the opposite angle, and is supplementary to the angle on either side of it. The four angles together make up four right angles.

16. This proves that the three angles together make up 180°.

17. 360°. The four angles together make up four right angles.

EXERCISES IX (*Pages* 36, 37).

1. 1″, 1·73″, 90°. **2.** 2·45″, 3·35″, 60°. **3.** 1·41″, 1″, 75°. **4.** 1·01″, 1·56″, 100°. **5.** 2·5 cms., 2·5 cms., 60°. **6.** 2·57 cms., 3·29 cms., 60°. **7.** $BC=4{\cdot}24$ cms., $B=C=45°$. **8.** $AB=2$ cms., $AC=3{\cdot}46$ cms., $B=60°$. **9.** $B=45°$, $AC=2{\cdot}45$ cms., $AB={\cdot}9$ cm. **10.** $AB=3{\cdot}02''$, $B=38{\cdot}8°$, $C=71{\cdot}2°$. **11.** $BC=1{\cdot}85$ cms; $B=C=72°$. **12.** $BC=2{\cdot}9$ cms., $A=24{\cdot}8°$, $B=35{\cdot}2°$. **13.** $AB={\cdot}66''$, $B=134{\cdot}2°$,

$C=10{\cdot}8^\circ$; or $AB=3{\cdot}44''$, $B=45{\cdot}8^\circ$, $C=99{\cdot}2^\circ$. **14.** $BC=3{\cdot}89$ cms., $A=34{\cdot}6^\circ$, $B=105{\cdot}4^\circ$; or $BC=6{\cdot}22$ cms., $A=65{\cdot}4^\circ$, $B=74{\cdot}6^\circ$. **15.** 1·5 cms. **16.** 1·89 cms. **17.** $AC=3{\cdot}78$ cms, $C=37{\cdot}4^\circ$. **18.** $AC=3{\cdot}46$ cms., $A=90^\circ$, $C=30^\circ$. **19.** 2·24″. **20.** 3 cms. **21.** 2·4″. **22.** 2·73″. **23.** Each 1·56″. **24.** 2·93 cms. **25.** 2·6″. **26.** 1·39″. **27.** 1·46″. **28.** 5·43 cms.

EXERCISES X (*Page* 40).

1. 1·22″. **2.** 1·75″. **3.** 125·9°. **4.** 40·9°. **5.** 75°. **6.** 2·13″. **7.** ·35″. **8.** 27·6 mms. **9.** 28·1 mms. **10.** 3·08″, 54°. **11.** 33·6 mms. **12.** 4·47″, 26·6°.

EXERCISES XI (*Pages* 43, 44).

1. Any line joining a point on the first perpendicular to a point on the third is bisected by the second perpendicular.

2. 40°, 90°, ·77″, ·77″.

4. (i) They are equal; (ii) they are equal; (iii) they are supplementary. Every angle in the figure measures either 42° or 138°.

5. If a line is at right angles to a second line it is also at right angles to any line which is parallel to the second line.

6. ·75″, ·75″, 1·16″, 1·16″. **7.** 3·46 cms. **8.** 2·4 cms.

9. 1·44″. **10.** 4·33 cms., 2·60 cms., 1·86 cms. **11.** 2·17″

12. 1·8″. **13.** 3·33 cms. **14.** 1″, 2·22″. **15.** 1·18″.

16. 1·70″. **17.** 1·2″, ·53″. **18.** 1·2″, 1·2″, ·69″, ·69″.

19. 1·40″, 1·40″, ·98″, ·98″. **20.** 1·99″. **21.** 1·44 cms.

EXERCISES XII (*Pages* 46, 47).

1. (i) Opposite sides are equal; (ii) opposite angles are equal, and adjacent angles are supplementary; (iii) the diagonals bisect each other.

2. (i) Opposite sides are equal; (ii) the diagonals are equal and bisect each other.

3. A rhombus. (i) Opposite angles are equal, and adjacent angles are supplementary; (ii) the diagonals are at right angles and bisect each other.

4. A square. The diagonals are at right angles, are equal, and bisect one another.

EXERCISES XIII (*Pages* 47–50).

1. Each diagonal measures 5 cms., and each part measures 2·5 cms. The diagonals of a rectangle are equal and bisect each other.

2. The diagonals measure 2″ and 3·46″; the parts measure 1″ and 1·73″; the diagonals intersect at right angles.

3. 130°, 50°, 130°. **4.** 4·24″; ·75″; ·75″; the base is bisected.

5. Each diagonal measures 2·83″, each angle 90°, and each part 1·41″; the diagonals are equal and bisect each other at right angles.

6. 180°. **7.** Two sides measure 4·63″, and the other two 2·12″; two angles measure 59·5°, and the other two 120·5°.

8. Angles at C and D each 53·1°; at A and B each 126·9°; two angles at intersection of diagonals measure 56·3°, and the other two measure 123·7°. **9.** 1·45″. **10.** 1·74″. **11.** $A=82{\cdot}8^\circ$, $B=110{\cdot}7^\circ$, $C=69{\cdot}3^\circ$, $D=97{\cdot}2^\circ$. **12.** Two sides measure 1·29″, and the other two 1·53″.

13. Each side measures 1·39″. **14.** Each angle measures 90°.

15. Each side measures 1·98″, and each angle 90°. **16.** 2·08″.

17. 2·87 cms. **18.** Each 2·83 cms. **19.** 3·24″, 3·24″, 108°, 108°.

20. 2·6″, 3″. **21.** 2·77″, 1·15″, 90°. **22.** 2·66″, 72°. **23.** 3·62″.

24. 90°, 60°.

EXERCISES XIV (*Pages* 52, 53).

1. $CF=DE=$ 2·92″. **2.** $AH=BG=$ 3·35″. **3.** $AQ=CP=$ 4·03″.

4. 2·8″. **5.** Each length is 1·29″. **6.** The points symmetrical to F and C are B and E; the lines symmetrical to FE and AB are BC and AF.

7. (i) Each of the two lines which bisects a pair of opposite sides. (ii) No; unless it is a rhombus, in which case each diagonal is an axis of symmetry.

8. (i) Each diagonal is an axis of symmetry; also each line which bisects a pair of opposite sides. (ii) Each of the three lines which passes through an angular point and bisects the opposite side is an axis of symmetry.

EXERCISES XV (*Pages* 56, 57).

1. (i) Six; (ii) twelve; (iii) eight. **2.** (i) AB, AD, AF; (ii) GC, GE, GH; (iii) EB, EF, EG. **3.** (i) CD, EF, GH; (ii) AF, BE, CG; (iii) AD, BC, FH. **4.** (i) BC, EG, FH; (ii) AF, BE, DH; (iii) AB, CD, GH. **5.** Four; four. **6.** Right side; front. **7.** $DE=BH$ $=$ 2·47″; each part of each diagonal $=$ 1·23″. **8.** $DE=CF=$ 2·47″; each part of each diagonal $=$ 1·23″. **9.** $DE=AG=$ 2·47″; each part of each diagonal $=$ 1·23″. **10.** The four diagonals of a rectangular block are equal and bisect one another. **11.** 2·60″.

EXERCISES XVI (*Page* 60).

1. (i) Six, four; (ii) eight, five; (iii) ten, six; (iv) sixteen, nine; (v) nine, six. **2.** Three. **3.** Four, four. **4.** Three, three. **5.** (*a*) ABC, DEF; (*b*) AD, BE, CF; (*c*) AB and DE, BC and EF, CA and FD. **8.** 1·4″.

EXERCISES XVII (*Page* 63).

1. (*a*) A circle; (*b*) a circle; (*c*) a cylinder; (*d*) the curved surface of the cylinder. **2.** (*a*) A circle; (*b*) a cone; (*c*) the curved surface of the cone. **3.** A sphere. **4.** A circle equal to the base of the cylinder. **5.** A circle smaller than the base of the cone.

6. (*a*) A circle whose radius is equal to that of the sphere; (*b*) a circle whose radius is less than that of the sphere.

EXERCISES XVIII (*Pages* 69, 70).

6. Space, position, distance. **10.** (*a*) Incomplete; (*b*) redundant; (*c*) redundant; (*d*) incomplete. **11.** (*a*) A very small particle has size, a point has not; (*b*) the path traced out by a moving particle has breadth and thickness; (*c*) any body has position; (*d*) a row of points is not continuous; (*e*) any body has length; (*f*) a "broken line" is neither straight nor curved.

EXERCISES XIX (*Pages* 75, 76).

1. The first five. **2.** Cylinder and cone. **3.** (*a*) AD; (*b*) BE; (*c*) DH. **4.** (*a*) VB; (*b*) AD. **5.** Edges FA, FE, FH; faces $FABE$, $FADH$, $FEGH$. **6.** Three edges; three faces. **7.** Four edges; four faces. **8.** Edges EB, ED, EF; faces $EBAD$, $EBCF$, EDF; faces $DABE$, $DACF$. **9.** A circle. **10.** A circle. **11.** If the sheet is flat, when it is dipped into the water the water line is always straight; if the sheet is bent round into a cylinder, the water line may be straight or curved. **12.** Straight. **13.** (i) Any layer has thickness; (ii) st. lines can be drawn on some curved surfaces; (iii) a plane figure may have curved boundaries; (iv) an arc of a circle would also satisfy this test. **14.** (i) $A=\frac{1}{2}E$; (ii) $A=E$; (iii) $A=E$; (iv) $A>E$; (v) $A<E$; (vi) no result.

EXERCISES XX (*Page* 81).

1. (i) 7·5°; (ii) 6·75°; (iii) 12·6°; (iv) 25·5°. **2.** (i) 4R, 360°; (ii) $\frac{2}{3}$R, 60°; (iii) 8R, 720°; (iv) 1·6R, 144°; (v) 5·6R, 504°; (vi) 10R, 900°. **3.** (i) Nine times; (ii) five times; (iii) three times. **4.** (i) 135°; (ii) 225°; (iii) 117°; (iv) 108°; (v) 212°; (vi) 207°. **5.** (i) *OW*; (ii) *OD*; (iii) *ON*; (iv) *ON*; (v) *OW*; (vi) *DO* produced; (vii) *OE*; (viii) *OW*. **6.** (i) 110°; (ii) 135°; (iii) 172°; (iv) 180°; (v) 120°. **7.** *W*. **8.** 20° *S*. of *W*. **9.** 20° *W*. of *N*. **10.** (i) 0; (ii) 2R; (iii) R; (iv) R; (v) $\frac{5}{3}$ R; (vi) $\frac{4}{3}$ R. **11.** (i) 75°; (ii) 75°; (iii) 165°; (iv) 15°. **12.** (i) 120°; (ii) 1·5°.

EXERCISES XXI (*Page* 86).

1. 142°. **2.** 45°. **3.** 145°. **4.** $\beta=125°$, $\gamma=55°$, $\delta=125°$. **5.** 210°. **6.** $\alpha=60°$, $\beta=120°$, $\gamma=60°$, $\delta=120°$. **7.** 90°. **8.** 90°.

EXERCISES XXII (*Pages* 96, 97).

1. δ, λ, σ. **2.** α, γ, θ, μ. **3.** $\gamma=\theta=\mu=72°$, $\beta=\delta=\lambda=\sigma=108°$. **4** $\mu=45°$, $\sigma=135°$. **5.** 130°, 80°. **6.** $\angle A=70°$, $\angle B=48°$, $\angle C=62°$. **7.** (i) μ (equal); (ii) α, θ (equal); (iii) β, δ, λ, σ (supplementary). **8.** (i) The angle might be zero; (ii) defining "in a circle"—the right angle must be defined before the term perpendicular is defined; (iii) straight lines which do not intersect are not parallel unless they are in the same plane. **9.** (i) If a man can benefit others he is rich—not necessarily true; (ii) if a line is not straight it is bent—true; (iii) if the moon cannot be seen the night is cloudy—not necessarily true; (iv) if a man is a good cricketer he has made a "century" in a first-class match—not necessarily true; (v) if two lines form an angle they meet—true.

EXERCISES XXIV (*Page* 103).

1. 238 yds. **2.** 34·8 cms., 25·6 cms. **3.** 7 ft. 4·7 in.
4. 57·8 ft. **5.** 108 miles *S*. **6.** 40° to the right.
7. 19·1° *S*. of *W*. **8.** 67 miles, 33·4° *N*. of *E*. **9.** 10·4 miles.

EXERCISES XXV (*Page* 112).

1. 65°. **2.** 67°. **3.** 60°. **4.** 50°, 50°, 80°. **5.** 72°, 72°, 36°.
6. 360°. **7.** (i) 6R; (ii) 8R; (iii) 10 R; (iv) 12 R. **8.** (i) 72°,
(ii) 108°. **9.** 135°.

EXERCISES XXXIV (*Pages* 132, 133).

1. 45°. **2.** 105°, 75°. **3.** 55°, 25°. **4.** 90°. **5.** 35°, 65°, 80°.
6. 25°. **7.** 120°. **8.** 36°, 72°, 108°, 144°. **9.** Eight.
10. Eight. **11.** 110°. **12.** Elevation = 25°. **13.** Elevation = 10°.
14. 23·4 ft. **15.** 270 ft. **16.** 16·1°. **17.** 52·9 ft. **18.** 28·6 yds.
19. 28·1 ft. **20.** 48·4 ft. **21.** 693 yds.

EXERCISES XXXVII (*Pages* 144–146).†

10. T.7, C.3. **13.** Pm.5. **14.** T.6. **18.** T.1, P.2.
19. Draw $AN \parallel BC$; P.1, T.2. **20.** 6 cms. **21.** 4·2″.
22. 6·56″. **23.** 3·71 cms. **24.** 2″. **25.** Each 5·83″. **26.** 38·7″.
27. 1·73″. **28.** 2·57″; 62°. **29.** 3·26 cms. **30.** 138·2°.
31. 2·83 cms.

EXERCISES XXXVIII (*Page* 149).

1. 1·37″. **2.** 4·77 cms. **3.** 1·22″. **4.** 1·5 cms. **5.** 1·67″.
6. 1·41″. **7.** ·69″. **8.** 1·51″. **9.** 1·72″ or ·33″.
10. 6·22 cms., or 3·89 cms. **11.** 2·44″. **12.** 2·13″.

EXERCISES XXXIX (*Pages* 155–157).

11. T.1, T.2. **12.** P.2*c*, T.2, T.2*c*. **13.** C.1, C.2, C.3.
15. 3·94 cms. **16.** ·5″. **17.** 1″. **18.** 1·8″. **19.** ·8″.
20. 2″, 2·24″, 2·45″. **21.** ·12 mile. **22.** ·385 mile.
23. 2·6 miles. **24.** 2 furlongs.

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† These hints do not include all the Propositions required to solve the riders.

EXERCISES XLI (*Pages* 162, 163).

1. A straight line; 1·25″, 1·14″, 1·14″, 1·25″. **2.** A straight line; ·89″, 1·79″. **3.** A straight line; *L*, ·85″, ·15″; *M*, 1·35″, ·65″. **4.** [One branch of a rectangular hyperbola]; ·98″, 1·6″. **5.** An ellipse; 1·88″, 1·63″, 1·18″. **6.** A hyperbola; ·8″, 1·11″, 1·44″. **7.** The line *AB*, *not* produced in either direction. **8.** The two lines obtained by producing *AB* in both directions. **9.** A circle of diameter 1·33″. **10.** 4·2, 4·5, 7·2 tenth-inches.

EXERCISES XLII (*Page* 165).

1. 1″, 2·22″. **2.** 1·99″. **3.** 1·70″. **4.** 1·2″, 1·2″, ·69″, ·69″. **5.** 8·5, 5·3 tenth-inches. **6.** 9·5, 2·5 tenth-inches.

EXERCISES XLIII (*Pages* 167–169).

1. Two, one, or none. **2.** Four. **3.** When $X \perp AB$ and does not bisect *AB*. **4.** Two or one. **5.** When the three points are collinear. **6.** When the three lines are parallel. **7.** Show that the intersection of two of these lines lies on the third. **8.** See 7. **9.** See 7. **14.** 1·56″. **15.** 2·16 cms. **16.** 1·5 cms. **17.** 1 cm. **18.** 2·02 cms. **19.** 1·54″. **20.** 3″. **21.** ·95″. **22.** 1·49″. **23.** 4·48″ or ·32″. **24.** A straight line ∥ to the bisector of $\angle XOY$. **25.** A straight line. **26.** 1″, ·5″. **27.** 3·8, 6·2 tenth-inches. **28.** 1·73″. **29.** (i) 1·81″, (ii) 1·76″. **30.** 3·6″, 1·4″. **31.** ·68″, 1·48″.

EXERCISES XLIV (*Page* 172).

1. *A*, 24; *B*, 36; *C*, 12; *D*, 14; *E*, 24; *F*, 50. **2.** $A=E$. **3.** $B=A+C$; $B=C+E$; $F=B+D$. **4.** (i) Three times; (ii) twice. **5.** $p \times q$.

EXERCISES XLV (*Page* 175).

1. ·24, ·36, ·12, ·14, ·24, ·5 sq. in. **2.** 1·73 sq. in. **3.** 2·48 sq. in. **4.** 2·01 sq. in. **5.** 4·52 sq. in. **6.** 7·07 sq. in.; 28·28 sq. in. **7.** In each case 3·14. Area of $\odot = 3{\cdot}14 \times$ area of the square on the radius. **8.** 1·57, 6·47 sq. in. **9.** ·81 sq. in. **10.** 9. **11.** 144. **12.** 64.

EXERCISES XLVI (*Pages* 181, 182).

1. (i) 7·35 sq. in.; (ii) 7·99 sq. cms.; (iii) ·0000001 sq. in.; (iv) ·9635 sq. cm. **2.** (i) 4″; (ii) ·2″; (iii) ·002″; (iv) 1 cm. **3.** (i) 5″; (ii) 50″; (iii) 4·9 cms.; (iv) ·3906 cm. **4.** 625 sq. yds. **5.** 629 sq. miles. **6.** 160 ft. **7.** 125·7 miles. **8.** 64 sq. yds. **9.** (i) 2·5 sq. in.; (ii) 2·5 sq. in.; (iii) 3·75 sq. in.; (iv) 3·25 sq. in.; (v) 4·25 sq. in.

EXERCISES XLVIII (*Page* 187).

1. 5·44 sq. in. **2.** 9 92 sq. cms. **3.** 4·96 sq. in. **4.** 9·14 sq. cms. **5.** ·97, 1·50, 5·33, 1·33 sq. in. **6.** 3·46, 1·53 sq. in.

EXERCISES XLIX (*Pages* 195–197).

1. Ar.5. **3.** Ar.5. **5.** Ques.4.; Pm.4*c*. **10.** Ar.1. **11.** Ar.3. **14.** Ar.3.; Ques.9. **15.** Ar.3; Ques.9. **18.** T.2; T.2*c*; P.2*c*. **19.** Ar.5; Ar.5*c*. **20.** Ar.5; Ar.5*c*. **21.** Ar.5. **22.** Ques.21. **23.** C.2; Pm.7. **24** L.5. **25.** Ar.5; L.5. **26.** Pm.1; Pm.2; Ar.2. **27.** Ar.3. **28** Use Ques. 27. **29.** Use Ques. 27.

EXERCISES L (*Page* 198).

1. 290 sq. in. **2.** 7·44 sq. miles. **3.** 707 sq. yds. **4.** 14·4 sq. yds. **5.** 7·81 sq. ft. **6.** 52·4 sq. ft. **7.** ·0000433 sq. in. **8.** ·00785 sq. in. **9.** $\frac{1}{5}$; $\frac{1}{25}$. **10.** $\frac{1}{3}$; $\frac{1}{3}$; $\frac{1}{9}$.

EXERCISES LI (*Page* 201).

1. 1·64″, ·67″, 2·01″. **2.** 4 cms., 5 cms. **3.** 4·14 sq. in. **4.** 6·63 sq. in. **5.** 9·39 cms. **6.** 10·85 cms. **7.** 3″. **8.** 76· 9 sq. cms.

EXERCISES LII (*Pages* 203–204).

1. 16 1°. **2.** 131·4° or 48·6°. **3.** 1·67″. **4.** 1·54″. **5.** 5·98 cms. **6.** 2·2″. **7.** 2·41″. **8.** 2·4″. **9.** 3″. **10.** 1·6″. **11.** 5·31 cms. **12.** 3·69, 7·67 cms.

EXERCISES LIII (Pages 206, 207).

3. Ar.5. **6.** Ques. 3. **12.** When BG passes neither between A and F nor between C and D. **14.** § 149. **15.** Use Ques. 14. **16.** Use Ques. 14. **17.** Any line through intersection of diagonals bisects the area. **18.** Compare § 150. **19.** Compare § 150. **20.** § 149, Ar.5. **23.** Intersection of two medians.

EXERCISES LIV (Page 208).

1. 25·9. **2.** 1·11. **3.** 6·72. **4.** 1·77. **5.** 1·92. **6.** 7·82. **7.** 2·65. **8.** 1·74.

EXERCISES LVI (Pages 216–219).

4. $5a^2=(2a)^2+a^2$. **12.** Sq. on BE = sq. on AE + 4 sq. on AF. **13.** Pm.4. **17.** (*c*) T.1. **18.** Sq. on perp. = sq. on side − sq. on half-side ; etc. **22.** 8·06″. **23.** 2·06 cms. **24.** 21·5 miles. **25.** 2·24″. **26.** 9·17 cms. **27.** 14·3 miles. **28.** 23·3 miles. **29.** 6ft. 11in. **30.** 83·1 miles. **31.** 4·58″. **32.** 2·65″. **33.** Thirteen ; 3,4,5. **34.** 3·2″, 1·8″, 2·4″, 1·92″, 2·56″, 1·44″. **35.** 9·37″; 32·8 sq. in. **36.** 10·4 yds. ; 62·4 sq. yds. **37.** 1·41″. **38.** 9·54″ ; 16″. **39.** 5·2″ ; 7·8 sq. in. **40.** 3·43″ ; 1·9 sq. in. **41.** 4·03″ ; 3·5 sq. in. ; 1·74″. **42.** 2·83″ ; 2·83″ ; 2·83 sq. in ; 1·89″. **43.** Let the equal sides measure a. **44.** Let hypotenuse measure $2a$. **45.** 6·45 cms. **46.** 4·9 mm. **47.** 3·16″. **48.** 34·8 mm. **49.** 3·6. **50.** 2·24. **51.** 6·4. **52.** 5·39. **53.** 4·8. **54.** 5·74. **55.** 6·56. **56.** 12·2. **57.** 1·28. **58.** 22·8. **59.** 1·12. **60.** 4·36.

EXERCISES LVII (Pages 221, 222).

1. (i) Equal and equally inclined to the axis ; (ii) Bisected at right angles by the axis. **2.** (i) Concurrent ; (ii) Concurrent. **3.** The pricks go through the circumference of the other semicircle ; any circle is symmetrical about any diameter.

EXERCISES LVIII (*Pages* 227, 228).

4. Sy.2, Sy.1. **6.** The join of the centres. When the circles are concentric. **11.** P.2*c*, Sy.3. **14.** **A, M, O, X** are symmetrical about "vertical" axes; **D, O, X** are symmetrical about "horizontal" axes; **O, X, Z** are symmetrical about points.

EXERCISES LIX (*Page* 231).

1. 90°; the diameter which bisects a chord is $\perp$ to it. **2.** The diameter through the point of contact of a tangent is $\perp$ to it. **3.** This line passes through the point of contact; draw a line through the given point, $\perp$ to the radius to that point. **4.** The locus of the mid-points of a system of parallel chords is the diameter $\perp$ to these chords. **5.** This diameter meets the curve at the points of contact of the tangents parallel to the chords. **6.** (i) 100°; (ii) 94°; (iii) 91°; (iv) 90·1°; (v) 90·001°; the tangent at C is $\perp$ to radius OC. **7.** In any circle the locus of the mid-points of a system of equal chords is a concentric circle. **8.** 2″. **9.** The greater.

EXERCISES LXIV (*Pages* 247–250).

18. Pm.1; Ch.1. **19.** Tn.2. **21.** Tn. 3. **22.** Tn.3. **23.** Tn.3; A.2*c*. **24.** Tn.3. **25.** C.2; Tn.2. **26.** A circle of radius 1·6″, concentric with the fixed circle. **27.** A circle of radius 2·2″, concentric with the fixed circle. **28.** 60°. **29.** 120°. **30.** 55°, 35°, 55°. **31.** 10°, 85°, 5°. **32.** $(90-\frac{1}{2}x)^\circ$, $\frac{1}{2}x^\circ$, $(90-\frac{1}{2}x)^\circ$. **33.** 5″, 6″, 7″. **34.** 5″, 5″, 4″. **35.** 140°, 125°, 95°. **36.** $\frac{1}{2}x^\circ$. **37.** 4·58″. **38.** 2·65″. **39.** 1·41″. **40.** 9·54″, 16″. **41.** 2·83″, 2·83″, 2·83 sq. inches, 1·89″. **42.** The larger chord, by ·2″. **43.** 2·24″. **44.** 11·6″. **45.** 35·7″. **46.** 29° in each case; the results are equal because the three cases are the same figure drawn to different scales. **47.** Two; viz. AB and the perpendicular diameter. **48.** Two; viz. AB and the perpendicular diameter. **49.** 3·82″. **50.** 2·67 cms. **51.** 1·6″. **52.** ·68 cm. **53.** ·78″.

EXERCISES LXV (*Pages* 252, 253).

1. Any angle in a semicircle is a right angle. **3.** (*a*) They are acute; (*b*) they are obtuse; (*c*) they are equal; (*d*) they are supplementary; (*e*) the first is double the second; (*f*) the first is double the second. **4.** A circle on diameter *AB*. **5.** (i) Two segments, one on each side of base *AB*; (ii) two segments, one on each side of base *AB*; (iii) this angle (60°) is double the angle in the larger segment (30°), and therefore lies on the circumference of the smaller segment. **6.** The angles between chord and tangent are equal to the angles in *alternate* segments.

EXERCISES LXVIII (*Page* 265).

1. $\frac{1}{4}$. **2.** $\frac{1}{3}$. **3.** 40°; 20°. **4.** 260°; 130°. **5.** $\frac{1}{6}$.
6. (i) $2x°$, $x°$; (ii) $360° - 2x°$, $180° - x°$.

EXERCISES LXX (*Pages* 268–270).

1. $\frac{x}{360}$. **2.** $\frac{x}{180}$. **3.** 7·33″, 36·67″. **4.** 62·3″; 1·59″.
5. (i) $\frac{7}{44}$; (ii) 57·3°; (iii) 28·6°. **6.** 88″; 720°. **7.** 7 [illegible]. **8.** 4·19″.
9. 140°. **10.** 30°. **11.** 10°, 50°. **12.** 20°, 50°. **13.** 120°.
14. 8″, 12″, 16″. **15.** 40°, 60°, 80°. **16.** 90°, 80°. **17.** ·71″.
18. 1·5″. **19.** 2·89″. **20.** 3·39″. **21.** 1·87″. **22.** 14·2 ft.
23. 2·93 ft. **24.** 5·8 cms., 5·8 cms., 8·04 cms.

EXERCISES LXXI (*Pages* 274–276).

27. AC.3. **28.** AC.2. **29.** AC.5*c*. **30.** T.1.
31. AC.3. **33.** AC.1. **34.** T.1. **35.** AC.3. **36.** AC.5.
37. P.2; T.1; Pm.2. **38.** T.1; AC.2. **39.** AC.6.
40. AC.3; AC.3*c*. **41.** T.1; AC.2. **42.** Ch.4; Ch.5.

EXERCISES LXXII (*Page* 278).

1. 4·47 cms. **2.** 3·65″. **3.** ·29″. **4.** 1·66 cms.

EXERCISES LXXIII (*Pages* 284, 285).

1. 2·83″. **2.** 2·88″. **3.** 4·58 cms. **4.** 2·18″.
5. (i) None ; (ii) two ; (iii) none ; (iv) four. **6.** 1·2 cms., 1·2 cms.
7. ·66, 1·51, 3 3, 1·98, 1·42 cms. **8.** ·29″, ·58″, ·87″.
9. 1, 2·5, 2, 3, 6 cms. **10.** 7·57″

EXERCISES LXXIV (*Page* 287).

1. 2·22″, ·58″. **2.** 1·58″. **3.** 5·61, 2·14 cms.
4. 4·61, 2·16 cms. **5.** 1″.

EXERCISES LXXV (*Pages* 292–298).

10. Eight. **16.** § 209. **17.** AC.1. **18.** § 200.
19. T.3 ; Ch.6. **20.** § 209. **21.** Ch.4, § 202. **22.** T.1 ; T.2.
23. Ch.4. **24.** Ch.4 ; Ch.5 ; § 202. **25.** C.1. **26.** T.1 ; Tn.2.
27. § 206 ; Tn.3*c*. **33.** ⊙ on diam. *BC*. **36.** § 209. **37.** § 209.
43. § 203. **44.** 70°. **45.** 19·1. **46.** 1·11 miles. **47.** 8·51 ft.
48. ·65″, 2·07″. **49.** ·5″, 1·5″. **50.** 1·96″.
51. 3·91, ·91 cms. **52.** 3·46 cms. **53.** 1·23″, 1·39″, 1·5″.
54. 4 cms., 5·14 cms., 7·52 cms. **55.** 1·73″. **56.** 7·96 cms., 6·22 cms.,
7·24 cms. **57.** 2·75″, 1·79″, 1·79″. **58.** ·62″. **59.** 1·43″.
60. 1·71″. **61.** 62°. **62.** 4·38 cms. **63.** ·67″. **64.** 1·66″.
65. 1·73″. **66.** 1·34″. **67.** ·69″. **68.** 3 cms. **69.** 2·83 cms.
70. 4·12 cms. **71.** Two at distance 1·66 cms. ; two, 4·33 cms. ;
four, 2·86 cms. **72.** 1·7″. **73.** 3·18″. **74.** 2 cms. **75.** 1·5 cms.
76. 2 cms. **77.** 1·95″. **78.** 5·88 cms. **79.** 73·7°.
80. ·49″, 2·04″. **81.** 4·68, 4·68, 3·6 cms. **82.** 3·02″, 1·79″, 1·98″.
83. 1·92″, 1·75″, 1·17″. **84.** 1·15″. **85.** 39·2° ; ·97 mile.

EXERCISES LXXVI (*Page* 300).

1. Four. **2.** Six ; more closely.

EXERCISES LXXVII (*Page* 304).

4. In $\triangle ABC$, if $\angle A = 2\angle B$, then BC is the base of one isosceles $\triangle$; if $\angle A = 3\angle B$, then AB is the base of one isosceles $\triangle$. **5.** On given base draw a segment to contain given angle, and complete the circle. **6.** § 214. **7.** § 214. **9.** Find the centre of the circle through the three given centres. **10.** Use the in-centre.

EXERCISES LXXX (*Page* 312).

3. Draw lines from A $\perp$ sides of $\triangle BCD$.

EXERCISES LXXXII (*Page* 317).

8. Prob. 4.; Indirect Method. **9.** Deduce from Prob. 4.

EXERCISES LXXXIII (*Pages* 320–329).

1. T.1. **2.** T.7. **3.** L.4. **4.** T.7. **5.** T.7; Pm.1. **6.** Use Theorem I, § 215. **7.** Use $\triangle GKC$ in Fig. 340. **8.** Apply T.4 to $\triangle$s GAB, GBC, GCA in Fig. 340. **9.** Pm.4; T.7. **10.** T.1. **11.** T.7. **12.** C.2; Pm.1. **13.** Use § 199, Ex. 1. **14.** T.1. **17.** Use Ques. 1. **18.** T.4. **19.** Use Ques. 9, Ex. *LXXIX.* **20.** T.1. **21.** C.4; Pm.1 (or T.1; Pm.1). **22.** Pm.2; T.7. **23.** C.1; T.2; Ar.5*c*. **24.** T.4. **25.** C.2; Pm.1. **26.** T.1. **27.** T.7. **28.** § 115. **29.** § 107. **30.** P.1*c*; T.2*c*; Pm.2. **31.** T.7; Ar.3; Ar.5. **32.** Pm.5. **33.** Pm.1. **34.** Pm.1. **35.** RT.2. **36.** RT.2. **37.** Ar.2. **38.** Pm.1; Ar.3. **39.** AC.6; RT.2. **40.** Cf. § 145. **41.** T.7. **42.** Ch.6. **43.** Tn.2. **44.** Tn.2. **45.** Tn.2. **46.** Tn.2. **47.** T.7. **48.** Sy.1; Sy.1*c*; Sy.2. **49.** L.4; Ch.1. **50.** L.4; Ch.1. **51.** Ch.7. **52.** C.2, Tn.2. **53.** Tn.2; AC.5. **54.** Ch.1. **55.** T.2, Tn.2. **56.** Tn.2. **57.** T.7; Tn.1. **58.** Tn.1. **59.** Tn. 1, and reductio ad absurdum. **60.** $\frac{1}{2}$R; Tn.2. **61.** T.2. **62.** T.7; AC.2*c*. **63.** Use Ques. 62. **64.** T.1; AC.2. **65.** A.1; AC.3*c*. **66.** AC.6. **67.** AC.3; AC.3*c*. **68.** AC.4; AC.6. **69.** AC.4. **70.** AC.3. **71.** T.1. **72.** T.1. **73.** AC.2; AC.2*c*. **74.** AC.2. **75.** T.2*c*; AC.2. **76.** C.4; Ch.1; AC.1. **77.** Tn.2; AC.5.

78. T.2; AC.5. **79.** AC.6. **80.** C.2; Tn.2. **81.** AC.4; AC.6.
82. C.1; AC.2. **83.** T.2*c*; AC.6. **84.** AC.2; AC.3.
85. T.1; AC.5. **86.** AC.2; AC.2*c*. **87.** AC.1; AC.2; AC.3.
94. Tn.3, Cor. **96.** Ch.4. **97.** The orthocentre. **98.** §§ 203, 204.
99. Draw the in-circle; Tn.2. **100.** Draw an ex-circle; Tn.2.
101. Ch.3. **102.** § 209. **103.** C.1. **104.** T.7. **105.** T.7.
106. L.5. **107.** L.6. **108.** L.6. **109.** AC.4. **110.** Cf. § 199, Ex.1.
111. RT.2. **112.** If in two circles equal chords subtend equal angles
at the circumference, then these circles are equal. **113.** The locus
is a straight line equally inclined to the two arms of the vertical
angle; use the figure of § 199, Ex. 1. **114.** T.3*c*; Ar.2. **115.** AC.4
116. Ch.6. **117.** AC.2. **118.** § 202. **119.** T.3*c*. **120.** Theorem 7,
Chap. *XIV*. **121.** Ch.6; and Ques. 3, Ex. *LXIV*. **122.** Ques. 9,
Ex. *XLIX*. **123.** Theorem 7, Chap. *XIV*; AC.2. **124.** Theorem
7, Chap. *XIV*; see the methods of argument used in Theorems 8, 9.
125, 126, 127, 128. See the methods used in Theorems 8, 9. **129.** C.2.
130. T.3. **131.** T.7. **132.** Prove $\angle CAP < \frac{1}{2} \angle A$, $\angle CAX > \frac{1}{2} \angle A$.
133. T.2; T.2*c*; T.7; or use Th. 3, Ch. *XIV*. **134.** A.1; L.5; AC.6.
135. Ar.5; reductio ad absurdum. **136.** Cf. Ques. 59.
137. Cf. Ques. 44. **138.** Tn.2. **139.** Cf. Ques. 44. **140.** Cf. Ques. 107.
141. Cf. Ques. 108. **142.** Prove $\angle RQC = \frac{1}{2} \angle B$ by T. 1.
143. Cf. Ques. 44. **144.** Call the third side the base; then prove
that the line through the vertex parallel to the base passes through a
fixed point which is the centre of the required circle: AC.6, Cor.
145. Let escribed circles touch sides BC, CA, AB at D, E, F; prove
$AE = BD$, $AF = CD$, $BF = CE$ (cf. Ques. 44); let perpendiculars through
D and E meet at O; then $OB^2 - OC^2 = BD^2 - CD^2$; etc.

EXERCISES LXXXV (*Pages* 332, 333).

1. Rect. $(P+Q, X+Y)$
$=$ rect. (P, X) + rect. (P, Y) + rect. (Q, X) + rect. (Q, Y).

2. Rect. (P, X) + rect. (P, Y) + rect. (P, Z) + rect. (Q, X) + rect. (Q, Y) + rect. (Q, Z).

3. Rect. $(P, X+Y+Z)$ = rect. (P, X) + rect. (P, Y) + rect. (P, Z); rect. $(X, X+Y)$ = sq. on X + rect. (X, Y); rect. $(X, 2X) = 2$ sq. on X; rect. $(X+Q, X+P)$ = sq. on X + rect. (P, X) + rect. (Q, X) + rect. (P, Q); sq. on $(X+Y)$ = sq. on X + sq. on Y + 2 rect. (X, Y); rect. $(2P, X+Y) = 2$ rect. $(P, X) + 2$ rect. (P, Y); sq. on $(2P) = 4$ sq. on P; sq. on $X = 4$ sq. on $(\frac{1}{2}X)$; sq. on $(3Q) = 9$ sq. on Q.

4. Rect. (P, X) − rect. (P, Y).

EXERCISES LXXXVI (*Page* 337).

5. 2 rect. (P, Q). **6.** 6 rect. (P, Q). **7.** 16 sq. on P.
8. Sq. on P + 2 sq. on Q + 3 rect. (P, Q).
9. 2 sq. on P + 2 sq. on Q + 5 rect. (P, Q).
10. Sq. on P + sq. on Q + 2 rect. (P, Q) + rect. (P, R) + rect. (Q, R).

EXERCISES LXXXVII (*Page* 340).

4. Sq. on P + sq. on Q + sq. on R + 2 rect. (P, Q) + 2 rect. (P, R) + 2 rect. (Q, R).

EXERCISES LXXXVIII (*Page* 341).

2. $(x+y)^2=x^2+2xy+y^2$. **3.** $(x-y)^2=x^2-2xy+y^2$.
4. $x^2-y^2=(x+y)(x-y)$. **5.** $(x+y)^2+(x-y)^2=2x^2+2y^2$.
$(x+y)^2-(x-y)^2=4xy$.

EXERCISES LXXXIX (*Page* (345).

8. $(a+b)^2=(a-b)^2+4ab$. **10.** R.6. **11.** R.6. **12.** R.2.

EXERCISES XC (*Pages* 346, 347).

1. $AZ=\cdot75$ sq. in., $BZ=2\cdot5$ sq. ins., $BX=2\cdot5$ sq. ins., $CX=1\cdot5$ sq. ins., $CY=1\cdot5$ sq. ins., $AY=\cdot75$ sq. in.
Thus $AY=AZ$, $BZ=BX$, $CX=CY$.

2. (i) $AB^2=CA^2+CB^2-(CA.CE)-(CB.CD)$;
(ii) $AB^2=CA^2+CB^2-2\,(CA.CE)$; (iii) $AB^2=CA^2+CB^2-2\,(CB.CD)$.

3. $AZ=6\cdot5$ sq. cms., $BZ=18\cdot5$ sq. cms., $BX=18\cdot5$ sq. cms., $CX=2\cdot5$ sq. cms., $CY=2\cdot5$ sq. cms., $AY=6\cdot5$ sq. cms.
Thus $AY=AZ$, $BZ=BX$, $CX=CY$.

4. (i) $AB^2=CA^2+CB^2+(CA.CE)+(CB.CD)$;
(ii) $AB^2=CA^2+CB^2+2\,(CA.CE)$; (iii) $AB^2=CA^2+CB^2+2\,(CB.CD)$.

5 (i) $BC^2=AB^2+AC^2-(AB.AF)-(AC.AE)$;
(ii) $BC^2=AB^2+AC^2-2\,(AB.AF)$; (iii) $BC^2=AB^2+AC^2-2\,(AC.AE)$.

EXERCISES XCI (*Pages* 356, 357).

1. (i) 31·6″, (ii) 40″. **2.** (i) 4·5 miles, (ii) 3·375 miles.
3. 0·75 yd., 0·5 yd. **4.** 3·5 yds., 2·625 yds. **5.** 2·71′, 4.2′, 14·7 sq. ft. **6.** 1·375′, 2·666′, 5·33 sq. ft. **7.** 5·15 miles.
8. 3·74″. **9.** 3·16″, 4·36″. **10.** 3·74′. **12.** Pm. 6, T.10.
13. T.10, R.3 (Cor. 2). **14.** R.3 (Cor. 2). **16.** (*a*) R.7 (p. 345); (*b*) R.8 (p. 345).

EXERCISES XCII (*Page* 357).

1. If P is a fixed point within a fixed circle, and if AB is a variable chord passing through P, then area of rect. $AP.PB$ is constant.

2. In the figure of Ques. 1, area of rect. $AP.PB$ is increased if P is moved nearer to the centre, and decreased if P is moved nearer to the ○ᶜᵉ.

3. Ch.1, RT.2. **4.** R.5. In the figure of Ques. 1 if AB is *any* chord through P, then area of rect. $AP.PB=r^2-d^2$.

5. If P is a fixed point without a fixed circle, and if AB is a variable chord passing through P, then area of rect. $PA.PB$ is constant. **6.** R.5. In the figure of Ques. 5 if AB is *any* chord through P, then area of rect. $PA.PB=d^2-r^2$. **7.** RT.2. If PAB is *any* secant through P, then $PT^2=PA.PB$. **8.** $PA.PB$ is constant, and becomes PT^2 when A and B move up to T.

EXERCISES XCIII (*Page* 360).

1. AC.2. **2.** AC.2*c*. **5.** $AP^2+AQ^2=2r^2$. **6.** Ch.1. **7.** 4″. **8.** 4·17″. **9.** 3·075″. **10.** $h+\frac{l^2}{4h}$. **11.** 375 sq. cms. **12.** 3·32 cms.

EXERCISES XCIV (*Page* 364).

3. AC.5. **4.** AC.3*c*. $BF.BA=BP.BE=BD.BC$. $CE.CA=CP.CF=CD.CB$. **8.** 1·3″. **9.** 2·05″. **10.** 256 sq. cms. **11.** 3·87 cms.

EXERCISES XCV (*Pages* 365, 366).

1. (i) 10·5; (ii) ·65; (iii) 15·4; (iv) 1·47. **2.** (i) 6·28; (ii) 3·39; (iii) 3·24; (iv) 4·72; (v) 5·2; (vi) 35·3; (vii) 2·17; (viii) 11·7. **3.** $\frac{3}{4}(a^2+b^2+c^2)$. **4.** 6·5 sq. ft. **5.** 4·52″, 3·71″, 2·11″. **6.** $(a^2+t^2)/2a$. **7.** $\sqrt{(a^2+b^2+6ab)}$. **8.** 6·25′. **9.** 2·31″. **10.** 5″, 3·265″. **11.** 5″, 3·265″. **13.** RC.3. **15.** RT.2, R.5. **16.** T.8. **17.** R.3, Cor.2. **20.** T.10. **21.** T.10.

EXERCISES XCVI (*Page* 368).

1. Tn.2. **2.** RC.3. **4.** RT.2. **5.** § 256, RT.2. **6.** Cf. Ques. 4 and 5. **8.** The radical axis of the two given circles. **9.** Find the radical centre. **10.** § 256. The point A is a circle of zero radius, and PA is a tangent to the circle. **11.** RC.2. **12.** A circle whose centre bisects the join of the given centres.

EXERCISES XCVII (*Pages* 370, 371).

1. 2·45 cms. **2.** ·95″. **4.** (i) 2·24 ; (ii) 3·16 ; (iii) 4·58 ; (iv) 5·92 ; (v) 8·37 ; (vi) 0·77 ; (vii) 1·52 ; (viii) 0·87. **5.** (i) 5·3 cms., 1·7 cms. ; (ii) 8·11 cms., 1·11 cms. **6.** (i) 1·71″, 0·29″ ; (ii) 2·22″, 0·22″. **8.** (i) 7·85, 1·15 ; (ii) 8·53, 0·47 ; (iii) 8·74, 0·26 ; (iv) $x=7·85$ or 1·15 ; (v) $x=-8·53$ or $-0·47$; (vi) $x=8·74$ or 0·26. **9.** (i) 10, 10 ; (ii) 17·1, 2·9 ; (iii) 15·5, 4·5. **10.** (i) 9 ; (ii) 12·25 ; (iii) $\frac{1}{4}x^2$. **11.** (i) 6 ; (ii) 3 ; (iii) $2\sqrt{x}$. **13.** (i) 6·4, 1·4 ; (ii) 5·7, 0·7 ; (iii) 5·42, 0·42 ; (iv) $x=6·4$ or $-1·4$; (v) $x=-5·7$ or 0·7 ; (vi) $x=5·42$ or $-0·42$. **14.** (i) 24·1, 4·1 ; (ii) 22·2, 2·2 ; (iii) 23, 3.

EXERCISES XCVIII (*Pages* 373, 374).

1. 4·94 cms., 3·06 cms. **2.** (i) 72°, 72°, 36° ; (ii) 72°, 72°, 36° ; (iii) 108°, 36°, 36°. In (i) and (ii) each of two angles is double the third ; in (iii) one angle is three times each of the others. **3.** 7·85 cms., 4·85 cms. **4.** (i) 108°, 36°, 36° ; (ii) 108°, 36°, 36° ; (iii) 72°, 72°, 36°. Cf. Answer to Ques. 2. **5.** 0·618″, 0·382″. **6.** If the segments are x and $a-x$, the equation is $x^2+ax-a^2=0$; this equation gives the pair of segments as either $\frac{1}{2}a(\sqrt{5}-1)$, $\frac{1}{2}a(3-\sqrt{5})$, or $-\frac{1}{2}a(\sqrt{5}+1)$, $\frac{1}{2}a(3+\sqrt{5})$. The first result is the answer required: the second gives the segments when the line is divided *externally* in medial section.

EXERCISES XCIX (*Page* 376).

1. 1·97 sq. ins. **2.** Each angle = 108°, each diagonal = 6 cms. Note that the pentagon is regular. **3.** Five. **4.** Ten. **5.** § 263. **6.** § 259.

EXERCISES C (*Page* 377).

1. 5·24, 0·76. **2.** 5·56, 1·44. **3.** 3·19, 0·31. **4.** 0·72, 0·28. **5.** 1·58, 0·42. **6.** 1·81, 0·69.

EXERCISES CI (*Page* 380).

1. 1·14, $-6·14$. **2.** 7·4, $-0·4$. **3.** $-6·19$, $-0·81$. **4.** $-2·85$, 0·35. **5.** The roots are imaginary $[-0·75 \pm 0·66.i]$*.

6. 2·42, −0·82. **7.** The roots are imaginary [$-0{\cdot}5 \pm 0{\cdot}87.i$]*.
8. The roots are imaginary [$1{\cdot}2 \pm 1{\cdot}25.i$]*. **9.** 15·5, 4·5.
10. 23, 3. **11.** The numbers are imaginary [$2{\cdot}5 \pm 2{\cdot}96.i$]*.
12. The numbers are imaginary [$15 \pm 8{\cdot}66.i$]*.

EXERCISES CII (*Page* 380).

1. 1·7 cms. **2.** 0·99″. **3.** 269 yds. **4.** 6·85′, 3·65′. **5.** 3·82 miles. **6.** 1·18 sq. ft. **8.** § 258. **9.** § 149; or use Fig. 390. **10.** Use Fig. 390. **11.** Use Fig. 390. **12.** §§ 257, 258. **13.** RT.2; $x=3{\cdot}2$, $y=1{\cdot}8$. **14.** RT.2; $x=3{\cdot}62$ or $1{\cdot}38$, $y=1{\cdot}38$ or $3{\cdot}62$. **15.** Use the properties of the half-equilateral-triangle.

EXERCISES CIII (*Pages* 381, 382).

1. The equilateral triangle and the square. **2.** No. **3.** 108°. No. **4.** 72°; 108°. Yes. **5.** $\frac{1}{n} \times 360°$; $180° - \frac{1}{n} \times 360°$. Yes. **7.** 1·76″, 2·85″. **8.** The trace in its new position fits exactly on to the original figure: hence the original pentagon has all its sides equal and all its angles equal, and is therefore regular. **9.** 4·34 cms. **10.** Each angle = 120°. No. The converse of Ques. 6 is not true.

EXERCISES CIV (*Page* 388).

8. C.2. **10.** 5·66 cms. **11.** 3·96″. **12.** 1·48″. **13.** 1·62″. **14.** 2·6″. **15.** 4·62 cms. **16.** 2·27″. **17.** 2·84″.

EXERCISES CV (*Pages* 392, 393).

4. Calculate $\angle AOD$. **5.** Calculate $\angle AOB$. **6.** Calculate the angles at K.

EXERCISES CVI (*Pages* 395–397).

1. Perimeter of inscribed equilateral triangle = $3 \times 5{\cdot}2'' = 15{\cdot}6''$; perimeter of circumscribed equilateral triangle = $3 \times 10{\cdot}4'' = 31{\cdot}2''$. **2.** Perimeter of inscribed square = $4 \times 4{\cdot}24'' = 16{\cdot}96''$; perimeter of circumscribed square = $4 \times 6'' = 24''$. **3.** Inscribed pentagon, $5 \times 3{\cdot}53'' = 17{\cdot}65''$; circumscribed pentagon, $5 \times 4{\cdot}36'' = 21{\cdot}8''$. **4.** Inscribed hexagon, $6 \times 3'' = 18''$; circumscribed hexagon, $6 \times 3{\cdot}46''$

$=20{\cdot}76''$. **5.** $7\times 2{\cdot}6''=18{\cdot}2''$; $7\times 2{\cdot}89''=20{\cdot}23''$. **6.** $8\times 2{\cdot}3''=18{\cdot}4''$; $8\times 2{\cdot}49''=19{\cdot}92''$. **7.** $9\times 2{\cdot}05''=18{\cdot}45''$; $9\times 2{\cdot}18''=19.62''$. **8.** $10\times 1{\cdot}85''=18{\cdot}5''$; $10\times 1{\cdot}95''=19{\cdot}5.''$ **9.** $12\times 1{\cdot}55''=18{\cdot}6''$; $12\times 1{\cdot}61''=19{\cdot}32''$. **10.** $18\times 1{\cdot}04''=18{\cdot}72''$; $18\times 1{\cdot}06''=19{\cdot}08''$. **11.** $30\times 0{\cdot}63''=18{\cdot}9''$; $30\times 0{\cdot}63''=18{\cdot}9''$. **12.** $\bigcirc^{ce}=18{\cdot}84''$; thus the perimeters of the inscribed and circumscribed 30-gons differ least from $\bigcirc^{ce}$. They appear both to exceed the $\bigcirc^{ce}$ by $0{\cdot}6''$: this is obviously wrong, as the perimeter of the inscribed 30-gon must be less than $\bigcirc^{ce}$. The question presupposes a greater degree of accuracy than can be obtained by these methods. [The following results are correct to four significant figures:—Inscribed 30-gon, $18{\cdot}82''$; circumference $18{\cdot}85''$; circumscribed 30-gon, $18{\cdot}92''$.]
13. $6{\cdot}88$ sq. inches. **14.** $9{\cdot}46$ sq. inches. **15.** $2{\cdot}84''$.
16. $1{\cdot}32''$. **17.** $1{\cdot}55''$. **18.** $0{\cdot}67''$. **19.** $1{\cdot}05''$.
20. $1{\cdot}15''$. **21.** $1{\cdot}08''$. **22.** $2r^2$. **23.** $2{\cdot}6r^2$. **24.** $0{\cdot}83a^2$.
25. πr^2. **26.** $27{\cdot}7$ sq. ins. **27.** 36 sq. ins. **28.** $41{\cdot}6$ sq. ins.
29. $43{\cdot}5$ sq. ins. **30.** $45{\cdot}9$ sq. ins. Given the perimeter of a regular polygon, the area increases with the number of sides, and is greatest when the number of sides is infinite, *i. e.* when the regular polygon becomes a circle. [Of *all* figures of given perimeter the circle contains the greatest area.] **31.** Ar.3. **32.** Pn.1. **33.** Pm.2.
34. Pn.4, AC.4. **35.** Pn.4, AC.4. **36.** T.2. **37.** Tn.2.
38. AC.6, Pn.2. **39.** AC.6, Pn.2. **40.** AC.5*c*. **42.** AC.6*c*.
43. Folding. **44.** Folding. **45.** Folding. **46.** § 263.

EXERCISES CVII (*Pages* 397–399).

1. R.5, RT.2. **2.** $AD^2=BD.DC+BC^2$. **3.** T.9. **4.** R.6, T.10.
5. RC.3. **6.** RC.3, RC.3*c*. **7.** § 258. **8.** RT.2, or R.6.
9. Use the result of Ques. 8. **10.** R.2, RC.3. **11.** AC.3*c*, AC.4, RC.3. **12.** AC.3*c*, R.2, RC.2. **13.** R.6, RC.3.
14. Cf. Ques. 31, Ex. *LXXXIII*. **15.** T.8 or T.9. **16.** Cf. Ques. 19, Ex. *XCV*. **17.** Cf. Ques. 16. **18.** T.10. **19.** L.8.
20. L.3. **21.** P.1, C.2, RC.2. **22.** $n=7, 9, 11, 13, 14, 17, 18, 19$.
23. C.3. **24.** L.3. **25.** If A, B, C are three consecutive vertices of the given hexagon, prove that $\triangle ABC$ is one-sixth of the hexagon, and is divided into three smaller triangles of equal area; etc.
26. If O is the centre, A, B two adjacent vertices of inscribed hexagon.

P the intermediate vertex of the circumscribed hexagon, and if OP cuts AB at K, prove that $\triangle OAP = \frac{4}{3} \triangle OAK$; etc. **27.** If A, B, C, D, E, F be the points of contact taken in order round $\bigcirc^{ce}$, prove that arc $AB =$ arc $CD =$ arc EF, and arc $BC =$ arc $DE =$ arc FA; etc. **28** Prove that the bisectors of the angles are concurrent. Cf. § 273 (1). **29.** If tangents at A, B meet at H, and tangents at A, C meet at K, prove $AH = HB = \frac{1}{2}HK$; etc. **30.** Ar. 3. **31.** AC.5, and symmetry. **32.** T.2*c*, AC.5*c*., RC.3. **33.** Cf. Ques. 31. **34.** T 9, R.2.

INDEX TO DEFINITIONS AND TERMS.

PAGE
Acute 78
Acute-angled triangle 104
Altitude of a segment 287
,, of a triangle 182
Analysis 193
Angle 17, 77, 79
,, in segment 251
,, of sector 251
Angular point 77
Apex 15
Arc 220
Area 171
Arm 77
Axiom 73
Axiomatic Theorem 74
Axis of cone 62
,, of cylinder 61
,, of symmetry 221

Base of segment 251
,, of triangle 14
Bisect 26

Centre 220
Centroid 306
Chord 220
Circle 220
Circumference 220
Circumscribe 246
Clockwise 79
Collinear 167
Commensurable 176
Common chord 243
,, tangent 279
Complement 195
Complementary 78
Conclusion 84
Concurrent 167
Concyclic 256
Cone 61
Congruence 112
Consecutive points 230
Constant quantities 191

PAGE
Contact 229
Converse 85
Convex polgyon 110
Corners 56
Corollary 64
Counter-clockwise 79
Cross 72
Cube 58
Curve 68
Cut 229
Cyclic 256
Cylinder 61

Data 84
Decagon 109
Definition 66
Degree 27
Depression 131
Described about 246
Diagonal 38
Diameter 220
Dimensions 71
Direct common tangent 279
Distance (between parallel straight lines) 144
,, (of point from straight line) 129
Dodecagon 109

Edges 56
Elements 126
Elevation 131
Equilateral 104
Escribed circle 284
Ex-circle 284
Ex-radius 284
Extreme and Mean Ratio 371

Faces 56
Fundamental concepts 65

Geometry 71
Graduations 6

PAGE
Heptagon 109
Hexagon 109
Horizontal 131
Hypotenuse 105
Hypothesis 84
Hypothetical construction 92

Image 221
Incommensurable 177
In-radius 283
Inscribe 246
Inscribed circle 283
Intersection 72
Isosceles 104

Join 72

Line 2
Locus 159

Maximum 313
Medial Section 371
Median 143
Method of exhaustion 130
Minimum 313

Nonagon 109

Obtuse 78
Obtuse-angled triangle 104
Octagon 109
Orthocentre 308
Orthogonal circles 312

Parallax 7
Parallel 87
Parallelogram 45
Pedal line 311
Pedal triangle 309
Pentagon 109
Perimeter 302
Perpendicular 78
Plane 72
Playfair's Axiom 91
Point 2
Points of the compass 80
Polygon 109
Prism 60

PAGE
Problem 64
Produce 4
Projections 211
Proposition 64
Protractor 28
Pyramid 58

Quadrilateral 38
Quindecagon 109

Radical Axis 366
,, Centre 368
Radius 220
Rectangle 45
Rectangular block 55
Redundant 66
Re-entrant 78
Rhombus 45
Rider 98
Right angle 77
Right-angled triangle 104

Scale 102
Scalene 104
Secant 229, 230
Sector 251
Segment of a circle 251
,, line 330
Semicircle 220
Sense 94
Solid figures 54
Space 65
Sphere 62
Square 45
Squared paper 170
Straight 67
Straight angle 77
Subtend 239
Superficies 71
Superposition 113
Supplementary 78
Surface 71
Symmetry 220, 227

Tangent 229, 230
Tetrahedron 57
Theorem 64
Touch 229, 242, 243

PAGE
Transversal 87
Transverse common tangent... 279
Trial and Error Method 393
Triangle 14

Units of area 172

PAGE
Vertex of cone 61
Vertex of triangle 14
Vertical 131
Vertically opposite angles ... 85

Wallace line 311
Wedge 59

New Books and New Editions published since July 1, 1907, are underlined.

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CONTENTS.

	PAGES
Education	2
Mathematics and Mechanics	3-5
Biology	5
Physics	6, 7
Chemistry	7
French	8
English Classics	9
English Text-Books	10
Philosophy	11
Modern History	11
Geography	12
Roman and Greek History	12
Latin and Greek Text-Books	13
Latin and Greek Classics	14, 15

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